Combinatorics, spring 2025, homework 5

April 6, 2025

Please explicitly state the principles and tools of combinatorics you use in your solutions. If applicable, construct a bijection between sets to support your reasoning. While it is not necessary to show your work in detail, ensure that you identify any algebraic equations you are solving and double-check that your solutions are accurate.

1. Let F_n be the number of subsets of $[n] = \{1, \ldots, n\}$ that contain no two consecutive elements for integer n. Find the recurrence that is satisfied by these numbers (do not forget about the base case), and write this recurrence in the form of the algebraic identity the generating function $F(s) = \sum_{n \geq 0} F_n s^n$ satisfies.

2. A function f is defined for all $n \geq 1$ by the relations

   - $f(1) = 1$
   - $f(2n) = f(n)$ for all $n \geq 1$
   - $f(2n+1) = f(n) + f(n+1)$ for all $n \geq 1$.

   Let $F(s) = \sum_{n \geq 1} f(n) s^{n-1}$. Show that

   $$F(s) = (1 + s + s^2) F(s^2).$$

   Check that

   $$\prod_{j \geq 0}^{\infty} \left(1 + s^{2^j} + s^{2^{j+1}}\right)$$

   (whatever this infinite product means) solves this equation.

3. Similarly to the problem we have seen in the tutorial, find the number of subsets of $\{1, \ldots, 2000\}$ such that the sum of their elements is divisible by 4.