# Combinatorics, winter 2024, homework 4

March 30, 2025

Please explicitly state the principles and tools of combinatorics you use in your solutions. If applicable, construct a bijection between sets to support your reasoning. While it is not necessary to show your work in detail, ensure that you identify any algebraic equations you are solving and double-check that your solutions are accurate.

1. Find the number of 7-digit numbers (no 0 in any position) such that they do not contain the sequence of adjacent digits 454 in this particular order.

2. Let $A_1, \ldots, A_n$ be finite sets. Prove, that the number of elements $x \in A_1 \cup \ldots \cup A_k$ with the following property

   *$x$ belongs to the intersection $A_{i_1} \cup \ldots A_{i_{n-2}}$ of some $n-2$ of these sets, but not to an interesection of any $n-1$ of these sets*

   is given by the following formula:

$$\sum_{\substack{J \subset \{1,\ldots,n\} \\ \#J = n-2}} \# \left( \bigcap_{j \in J} A_j \right) - (n-1) \sum_{\substack{J \subset \{1,\ldots,n\} \\ \#J = n-1}} \# \left( \bigcap_{j \in J} A_j \right) + \binom{n}{2} \# \left( \bigcap_{j \in \{1,\ldots,n\}} A_j \right).$$

3. Prove that for any $n \in \mathbb{N}$ we have

$$\sum_{d|n} \phi(d) = n,$$

   where the sum ranges over all the positive divisors of $n$, and $\phi \colon \mathbb{N} \to \mathbb{N}$ is the Euler's totient function. Hint: denote the right-hand part by $\psi(n)$. Prove that for $gcd(m, n) = 1$ we have $\psi(nm) = \psi(n)\psi(m)$, and then prove that for any prime $p$ and any natural number $d$ we have $\psi(p^d) = p^d$.