Combinatorics, winter 2024, homework 2

March 23, 2025

Please explicitly state the principles and tools of combinatorics you use in your solutions. If applicable, construct a bijection between sets to support your reasoning. While it is not necessary to show your work in detail, ensure that you identify any algebraic equations you are solving and double-check that your solutions are accurate.

1. A 10-store building is to be painted with some 4 different colors such that each store is painted with one color. How many ways are there to paint the building in the following four scenarios:

   - there are no restrictions, not every color needs to be used;
   - any two adjacent stores must be painted with different colors, not every color needs to be used;
   - all the colors must be used at least once;
   - there are exactly 4 stores painted with color 1, 3 stores painted with color 3, 2 stores painted with color 3, and 1 store painted with color 4.

2. Find the number of integer solutions to the equation

   $$x_1 + x_2 + x_3 + x_4 = 30$$

   in each of the following cases

   - $x_i \geq 0$ for each $i = 1, 2, 3, 4$;
   - $2 \leq x_1 \leq 7$ and $x_i \geq 0$ for each $i = 2, 3, 4$;
   - $x_1 \geq -5, x_2 \geq -1, x_3 \geq 1$, and $x_4 \geq 2$.

3. Show that for any $n \in \mathbb{N}$ the number

   $$\frac{(n^2)!}{(n!)^n}$$

   is integer.