Combinatorics, spring 2025, homework 6

April 20, 2025

Please explicitly state the principles and tools of combinatorics you use in your solutions. If applicable, construct a bijection between sets to support your reasoning. While it is not necessary to show your work in detail, ensure that you identify any algebraic equations you are solving and double-check that your solutions are accurate.

1. Find the inverse of the formal power series from $\mathbb{C}[[s]]$ for a series given by $\sum_{k=0}^{\infty}(k+1)s^k$. Express the answer in the form $\sum_{k=0}^{\infty} c_k s^k$:

2. Using the generating function for the Fibonacci numbers, prove the following identities:

   - $f_0 + f_1 + \cdots + f_n = f_{n+2} - 1$;
   - $f_0 + f_2 + \cdots + f_{2n} = f_{2n+1}$;
   - $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n} - 1$.

   Here $f_0 = f_1 = 1$.

3. Find the generating functions and explicit expressions for the sequence given by the recurrence relation:

$$a_{n+3} = 2a_{n+1} - a_n,\ a_0 = 1, a_1 = -1, a_2 = 2.$$

4. Let $A(s) = \sum_{k=0}^{\infty} a_k s^k$, $B(s) = \sum_{k=0}^{\infty} b_k s^k$, $C(s) = \sum_{k=0}^{\infty} c_k s^k$.

   - Express $A(s)$ via $B(s)$ if $b_n = \sum_{k=0}^{n} a_k$;
   - Express $C(s)$ via $A(s)$ and $B(s)$ if $c_n = \sum_{j+2k=n} a_j b_k$;
   - Express $C(s)$ via $A(s)$ and $B(s)$ if $c_n = \sum_{j+2k\leq n} a_j b_k$;