1. Let V be an $n-$dimensional vector space over a field $\mathbb{K}$. Which is the characteristic polynomial of the identity operator on V? Which is the characteristic polynomial of the zero operator? Which are their minimal polynomials?

2. (a) Find the characteristic polynomial, eigenvalues and eigenvectors of the following matrices, considering separately the case in which the coefficients are in $\mathbb{R}$ and in $\mathbb{C}$

(i) $\begin{pmatrix} 1 & 3 \\ -3 & -1 \end{pmatrix}$.

(ii) $\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}$.

(iii) $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$.

(iv) $\begin{pmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix}$.

(v) $\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ 4 & -8 & -2 \end{pmatrix}$.

(vi) $\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$.

(vii) $\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & a & 1 \end{pmatrix}$.

(viii) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(ix) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

In all cases, $a \in \mathbb{K}$ $\quad (\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ respectively).

(b) Interpret each of the previous matrices as the matrix of a linear operator $T : \mathbb{K}^n \to \mathbb{K}^n$, represented in the standard ordered basis of $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ respectevely). Find, if it is possible, a basis $\mathcal{B}$ such that $[T]_\mathcal{B}$ is diagonal, and the change of bases matrix.

3. Determine which matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $a, b \in \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ are diagonalizable.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x, x - y + z, -3x - 2z)$$

(a) Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that $|T|_\mathcal{B}$ is diagonal.

(b) Find $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ -3 & 0 & -2 \end{pmatrix}^n$, $\forall n \in \mathbb{N}$.

5. (a) Let $A \in \mathcal{M}_2(\mathbb{C})$ with real entries and such that $\begin{pmatrix} 1+i \\ 2-i \end{pmatrix}$ is an eigenvector of A of eigenvalue $1 + 3i$. Show that A is diagonalizable, find A, and find a basis of eigenvectors.

(b) Let $A \in \mathcal{M}_2(\mathbb{R})$ be such that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A of eigenvalue $\sqrt{2}$, and such that the characteristic polynomial has rational coefficients, i.e., $\chi_A(t) \in \mathbb{Q}[t]$. Is A diagonalizable? How many matrices satisfy these conditions?

6. Let A and B be $n \times n$ matrices over the field $\mathbb{K}$. Prove that if $(I - AB)$ is invertible, then $I - BA$ is invertible and

$$(I - BA)^{-1} = I + B(I - AB)^{-1}A.$$

7. Use the result of Exercise 6 to prove that, if A and B are $n \times n$ matrices over the field $\mathbb{K}$, then AB and BA have precisely the same characteristic values.

8. Let A be a $n \times n$ matrix over the field $\mathbb{K}$. Show that A and A^t have the same characteristic values. Show with an example that this is not true for characteristic vectors.

9. Let A be a diagonalizable $n \times n$ matrix over a field $\mathbb{K}$. Let $\lambda_1, \cdots, \lambda_n$ be the roots of its characteristic polynomial counted with multiplicity. Show that $\mathrm{tr}(A) = \sum \lambda_i$, $\det(A) = \prod \lambda_i$

10. (a) Let A be a 3×3 real matrix such that $\mathrm{tr}(A) = -4$. Find the eigenvalues of A knowing that the eigenvalues of $A^2 + 2A$ are $-1, 3, 8$.

(b) Let A be a 4×4 real matrix such that $\det(A) = 6$, and such that $1, -2$ are eigenvalues of A, and -4 is an eigenvalue of $A - 3I$. Find all the eigenvalues of A.

11. Let a, b and c be elements of a field $\mathbb{K}$, and let A be the following 3×3 matrix over $\mathbb{K}$:

$$A = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}.$$

Show that the characteristic polynomial of A is $x^3 - ax^2 - bx - c$ and that this is also its minimal polynomial.

12. Let A be the 4×4 real matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

Show that the characteristic polynomial of A is $x^2(x-1)^2$ and that it is also the minimal polynomial.

13. Let n be a positive integer, and let V be the space of polynomials over $\mathbb{R}$ spanned by $\{1, x, x^2, ..., x^n\}$. Let D be the differentiation operator on V. Which is the minimal polynomial for D?

14. Use the Hamilton-Cayley Theorem:

(a) Find $\frac{1}{2}A^4 + A^3 - 6A^2 - 10A + 8I_2$ for $A = \begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}$.

(b) Find $\begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}^n$, $\forall n \in \mathbb{N}$.