1. **(20 p.)** Let f be a linear operator on $\mathbb{K}^2$. Prove that any non-zero vector which is not a characteristic vector for f is a cyclic vector for f. Hence, prove that either f has a cyclic vector or f is a scalar multiple of the identity operator.

2. Define the linear operator f on $\mathbb{R}^7$ by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}.$$

(a) **(15 p.)** Show that f does not admit a cyclic vector.

(b) **(15 p.)** Show that $W = \operatorname{span}\{e_3, e_4 + e_6, e_5\}$ is not an f-admissible subspace.

3. Let f be a linear operator on $\mathbb{R}^9$ whose matrix representation in the standard basis is

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -3 & 3 & 2 & 0 & 0 & 0 & -2 & 0 & 3 \\ -2 & 2 & 0 & 2 & 0 & 0 & -2 & 0 & 2 \\ 1 & -1 & 0 & -1 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define $W = \operatorname{span}\{e_6,\ e_1 + e_9\}$.

(a) **(20 p.)** Show that W is an f-admissible subspace of $\mathbb{R}^9$.

(b) **(30 p.)** Using W determine two distinct cyclic decompositions of $\mathbb{R}^9$ associated with the linear operator f.