1. Consider the linear operator $f : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$f(x, y, z, t) = (-4x - y - 5z + 5t,\ 9x + 2y + 9z - 9t,\ 4x + 2y + 5z - 4t,\ x + y + z).$$

(a) **(25 p.)** Find linear operators F_1, F_2 and F_3 such that

- $f = \alpha_1 F_1 + \alpha_2 F_2 + \alpha_3 F_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$,
- $F_i^2 = F_i$ for $1 \leq i \leq 3$,
- $\mathrm{id}_{\mathbb{R}^4} = F_1 + F_2 + F_3$, and,
- $F_i F_j = 0$ for $1 \leq i, j \leq 3$ and $i \neq j$.

(b) **(10 p.)** Find a diagonalizable operator D and a nilpotent operator N such that

- $f = N + D$,
- $DN = ND$.

2. Consider the linear operator $f : \mathbb{R}^5 \to \mathbb{R}^5$ given by

$$f(x, y, z, u, v) = (z,\ -2x - 2y + 2u,\ x - u,\ z,\ -3x - 3y + 3u + v).$$

(a) **(25 p.)** Find f-invariant subspaces W_1, W_2 and W_3 such that

- $\mathbb{R}^5 = \bigoplus_{1 \leq i \leq 3} W_i$,
- $m_f = \prod_{1 \leq i \leq 3} m_{f|_{W_i}}$, and
- If $i \neq j$, then $m_{f|_{W_i}}$ and $m_{f|_{W_j}}$ are coprime.

(b) **(10 p.)** Find a cyclic vector for f.

3. (30 p.) Let f be a linear operator on $\mathbb{R}^8$ whose matrix representation in the standard basis is

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Find a vector v in $\mathbb{R}^8$ such that the minimal polynomial of v with respect to f coincides with the minimal polynomial of f, *i.e.*, $m_{f,v}(x) = m_f(x)$.