1. For the following pairs A, B, decide if there exists an invertible matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal. If the answer is positive, find such P.

(a) **(15 p.)** $A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

(b) **(15 p.)** $A = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{pmatrix}$.

2. Let A, $A' \in \mathbb{K}^{5\times 5}$ be the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(a) **(10 p.)** Prove that both matrices are nilpotent.

(b) **(15 p.)** Find bases $\mathcal{B}$ and $\mathcal{B}'$ for $\mathbb{R}_4[x]$ so that the matrix representation of the derivative operator with respect to $\mathcal{B}$ is A, and with respect to $\mathcal{B}'$ is A'.

(c) **(10 p.)** Show that A is similar to A'.

3. Consider $\mathbb{R}^4$ as a vector space over $\mathbb{R}$ and let

$$W_1 = \operatorname{span}\{e_1 - e_3, e_1 - e_2\}, \quad W_2 = \operatorname{span}\{e_3 + e_4\}, \quad W_3 = \operatorname{span}\{-e_1 + e_2 - e_3\},$$

where $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ is the standard basis for $\mathbb{R}^4$.

(a) **(5 p.)** Show that $\mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$.

(b) **(30 p.)** Find projections F_i on $T_i = \bigoplus_{\substack{1\le j\le 3 \\ j\ne i}} W_j$ along W_i for $i = 1, 2, 3$.