1) (30 pts) Let $\mathcal{B} = \left\{ \begin{pmatrix} 0 & 2 \\ -3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ 2 & -3 \end{pmatrix} \right\}$.

(a) Prove that $\mathcal{B}$ is a basis of $\mathrm{M}_2(\mathbb{R})$.

(b) Find the change of basis matrix from the canonical basis $\mathcal{C}$ to $\mathcal{B}$ and the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$.

(c) Find the coordinates of any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$ in $\mathcal{B}$.

(a) linear independent: $0 = a_1\begin{pmatrix} 0 & 2 \\ -3 & 3 \end{pmatrix} + a_2\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} + a_3\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} + a_4\begin{pmatrix} 4 & -2 \\ 2 & -3 \end{pmatrix}$

Then $0 = \begin{pmatrix} 2a_3+4a_4 & 2a_1-a_2+a_3-2a_4 \\ -3a_1+2a_2-2a_3+2a_4 & 3a_1+a_3-3a_4 \end{pmatrix} \Rightarrow \begin{cases} 2a_3+4a_4=0 \\ 2a_1-a_2+a_3-2a_4=0 \\ -3a_1+2a_2-2a_3+2a_4=0 \\ 3a_1+a_3-3a_4=0 \end{cases}$

$\Rightarrow \begin{cases} a_3=-2a_4 \\ 2a_1-a_2=4a_4 \\ -3a_1+2a_2=-6a_4 \\ 3a_1=5a_4 \end{cases} \Rightarrow \begin{cases} a_1=\frac{5}{3}a_4 \\ a_2=-\frac{2}{3}a_4 \\ a_3=-2a_4 \\ -5a_4-\frac{4}{3}a_4=-6a_4 \end{cases} \Rightarrow \begin{cases} a_1=0 \\ a_2=0 \\ a_3=0 \\ a_4=0 \end{cases}$

Thus B is l.i. and length of $B=4=\dim(M_2(\mathbb{R}))$.
Thus B is a basis

(b) $P_C^B = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 2 & -1 & 1 & -2 \\ -3 & 2 & -2 & 2 \\ 3 & 0 & 1 & -3 \end{pmatrix}$ → This is change basis from B to C

Since $C=\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

Then $P_B^C = (P_C^B)^{-1} = \begin{pmatrix} -1 & -10 & -5 & 2 \\ \frac{1}{2} & 3 & 2 & 0 \\ \frac{3}{2} & 12 & 6 & -2 \\ -\frac{1}{2} & -6 & -3 & 1 \end{pmatrix}$ → This is change basis from C to B

(c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_B = (P_C^B)^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}_C$

$= \begin{pmatrix} -1 & -10 & -5 & 2 \\ \frac{1}{2} & 3 & 2 & 0 \\ \frac{3}{2} & 12 & 6 & -2 \\ -\frac{1}{2} & -6 & -3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} -a-10b-5c+2d \\ \frac{1}{2}a+3b+2c \\ \frac{3}{2}a+12b+6c-2d \\ -\frac{1}{2}a-6b-3c+d \end{pmatrix}$

2) (40 pts) Let $T\colon \mathrm{M}_2(\mathbb{R}) \to \mathrm{P}_2(\mathbb{R})$ be the linear map given by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b+d)x^2 + cx + (b+2c)$.

(a) Find the matrix $[T]_{\mathcal{C}\mathcal{C}'}$ of T with respect to the canonical bases $\mathcal{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\mathcal{C}' = \{x^2, x, 1\}$ of $\mathrm{M}_2(\mathbb{R})$ and $\mathrm{P}_2(\mathbb{R})$ respectively.

(b) Find the matrix $[T]_{\mathcal{C}\mathcal{B}'}$ of T with respect to $\mathcal{C}$ and $\mathcal{B}' = \{x-1, 2x^2+x-1, 3\}$.

(c) Find the matrix $[T]_{\mathcal{B}\mathcal{B}'}$ of T with respect to $\mathcal{B} = \left\{ \begin{pmatrix} 0 & 2 \\ -3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ 2 & -3 \end{pmatrix} \right\}$ and $\mathcal{B}'$.

(d) Find $\mathrm{Ker}(T)$ using the matrix $[T]_{\mathcal{B}\mathcal{B}'}$.

(a) $T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = x^2$, $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = x^2+1$, $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = x+2$, $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = x^2$

Then $[T]_{CC'} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$

(b) $[T]_{CB'} = \left(P_{C'}^{B'}\right)^{-1} [T]_{CC'} \cdot P_C^C$ where $P_{C'}^{B'} = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}^{-1} [T]_{CC'} \cdot \mathrm{Id}$

$= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 1 & 0 \end{pmatrix}$

(c) $[T]_{BB'} = \left(P_{B'}^{B'}\right)^{-1} [T]_{CB'} \cdot P_C^B$ where $P_C^B = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 2 & -1 & 1 & -2 \\ -3 & 2 & -2 & 2 \\ 3 & 0 & 1 & -3 \end{pmatrix}$

$= \mathrm{Id} \cdot \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 4 \\ 2 & -1 & 1 & -2 \\ -3 & 2 & -2 & 2 \\ 3 & 0 & 1 & -3 \end{pmatrix}$

$= \begin{pmatrix} -\frac{11}{2} & \frac{5}{2} & -4 & \frac{5}{2} \\ \frac{5}{2} & -\frac{1}{2} & 2 & -\frac{1}{2} \\ -\frac{7}{3} & \frac{5}{3} & -\frac{5}{3} & \frac{4}{3} \end{pmatrix}$

(d) $\ker T = \ker [T]_{BB'} = \ker \begin{pmatrix} -\frac{11}{2} & \frac{5}{2} & -4 & \frac{5}{2} \\ \frac{5}{2} & -\frac{1}{2} & 2 & -\frac{1}{2} \\ -\frac{7}{3} & \frac{5}{3} & -\frac{5}{3} & \frac{4}{3} \end{pmatrix}$

$= \ker \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{7}{3} \end{pmatrix}$ Thus $x' = 2w'$, $y' = -\frac{1}{3}w'$, $z' = -\frac{7}{3}w'$

Then $\ker T = \langle (6, -1, -7, 3) \rangle$

Then $6\begin{pmatrix} 0 & 2 \\ -3 & 3 \end{pmatrix} - 1\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} - 7\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix} + 3\begin{pmatrix} 4 & -2 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

Thus $\ker T = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$

3) (30 pts) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $T(x, y, z) = (3x + 4y - z, x + y + z, -3x + 6y)$ and consider the bases of $\mathbb{R}^3$ given by $\mathcal{B} = \{(1, 1, 0), (0, 1, 1), (1, 0, 0)\}$ and $\mathcal{B}' = \{(1, -2, 1), (2, -3, 3), (-2, 2, -3)\}$.

(a) Find the matrix $[T]_{\mathcal{B}'\mathcal{B}'}$ of T with respect to the basis $\mathcal{B}'$.

(b) Write $[T]_{\mathcal{B}'\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}\mathcal{B}}P$ where P is an invertible matrix.

(c) Is T an isomorphism? If so, describe $T^{-1}(x, y, z)$.

(a) $[T]_{B'B'} = [T]_{B'} = (P_C^{B'})^{-1} [T]_C \cdot (P_C^{B'})$ where $P_C^{B'} = \begin{pmatrix} 1 & 2 & -2 \\ -2 & -3 & 2 \\ 1 & 3 & -3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & -2 \\ -2 & -3 & 2 \\ 1 & 3 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 4 & -1 \\ 1 & 1 & 1 \\ -3 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 \\ -2 & -3 & 2 \\ 1 & 3 & -3 \end{pmatrix}$

$= \begin{pmatrix} 12 & 21 & -21 \\ -6 & -14 & 19 \\ 3 & 1 & 6 \end{pmatrix}$

(b) Lets find $[T]_B$

$[T]_B = (P_C^{B})^{-1} [T]_C \cdot (P_C^{B})$ where $P_C^{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 4 & -1 \\ 1 & 1 & 1 \\ -3 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -1 & -4 & 4 \\ 3 & 6 & -3 \\ 8 & 7 & 1 \end{pmatrix}.$$

Then $[T]_{B'B'} = (P_B^{B'})^{-1} \cdot [T]_{BB} \cdot P_B^{B'}$

$$[T]_{B'B'} = (P_C^{B'})^{-1}[T]_C \cdot (P_C^{B'}) = (P_B^{B'})^{-1} \cdot (P_C^{B})^{-1}[T]_C \cdot (P_C^{B}) \cdot P_B^{B'}$$

Since $[T]_{B'B'} = [T]_{B'} = (P_C^{B'})^{-1}[T]_C \cdot (P_C^{B'})$ and $[T]_B = (P_C^{B})^{-1}[T]_C \cdot (P_C^{B})$

Then $(P_C^{B'}) = (P_C^{B}) \cdot P_B^{B'} \Rightarrow (P_C^{B})^{-1}(P_C^{B'}) = P_B^{B'}$

Then $P_B^{B'} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & -2 \\ -2 & -3 & 2 \\ 1 & 3 & -3 \end{pmatrix} = \begin{pmatrix} -3 & -6 & 5 \\ 1 & 3 & -3 \\ 4 & 8 & -7 \end{pmatrix}$

Thus

$$[T]_{B'B'} = \begin{pmatrix} 12 & 21 & -21 \\ -6 & -14 & 19 \\ 3 & 1 & 6 \end{pmatrix} = (P_B^{B'})^{-1} \begin{pmatrix} -1 & -4 & 4 \\ 3 & 6 & -3 \\ 8 & 7 & 1 \end{pmatrix} \cdot P_B^{B'}$$

where $P_B^{B'} = \begin{pmatrix} -3 & -6 & 5 \\ 1 & 3 & -3 \\ 4 & 8 & -7 \end{pmatrix}$

(c) Yes. Since $\dim V = \dim \mathbb{R}^3 = 3 = \dim W = \dim \mathbb{R}^3$

$[T]_C = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 1 & 1 \\ -3 & 6 & 0 \end{pmatrix}$. $[T^{-1}]_C \times [T]_C = Id$. Then $[T^{-1}]_C = [T]_C^{-1}$

Then $[T^{-1}]_C = \begin{pmatrix} \frac{2}{13} & \frac{2}{13} & -\frac{5}{39} \\ \frac{1}{13} & \frac{1}{13} & \frac{4}{39} \\ -\frac{3}{13} & \frac{10}{13} & \frac{1}{39} \end{pmatrix}$

Then $T^{-1}(x,y,z) = xT^{-1}(1,0,0) + yT^{-1}(0,1,0) + zT^{-1}(0,0,1)$

$$= x\left(\frac{2}{13}, \frac{1}{13}, -\frac{3}{13}\right) + y\left(\frac{2}{13}, \frac{1}{13}, \frac{10}{13}\right) + z\left(-\frac{5}{39}, \frac{4}{39}, \frac{1}{39}\right)$$

$$= \left(\frac{2}{13}x + \frac{2}{13}y - \frac{5}{39}z, \frac{1}{13}x + \frac{1}{13}y + \frac{4}{39}z, -\frac{3}{13}x + \frac{10}{13}y + \frac{1}{39}z\right)$$