Algebra A.

3.3. CLASS 1

USEFULL for what? The more you doubt, the more you learn.

Example: $f: R \to R^2$, $f(1) = (3,-1)$. can we deduce $f(a)$ for any $a \in R$?

$f: R \to R^2$ is a Function satisfying:

1) $f(1) = (3,-1)$

2) R and R^2 HAVE ALGEBLA STRUCTURES AND f behaves:

$R \times R \xrightarrow{\cdot} R$, $R \times R^2 \xrightarrow{*} R^2$.

$(a,b) \longrightarrow ab$. $(a,(x,y)) \longrightarrow (ax, ay)$.

$f(ab) = a \cdot f(b)$. $f(a) = (x_a, y_a)$ $f(ab) = a * f(b)$.

$(x_{ab}, y_{ab}) = a * (x_b, y_b) = (ax_b, ay_b)$.

$f(a) = f(a \cdot 1) = a * f(1) = a * (3,-1) = (3a, -a)$.

F. VECTOR SPACES: sets of vectors with two Algebraic STRUCTURES.

LINEAR MAPS: Functions Between F. v.s. That behave well The Algebraic ST.

FIELD = SET WITH THE ALGEBRA STRUCTURES.

Field.

Def: A field is a nonempty set F with Two ALGEBRA STRUCTURES.

$+: F \times F \to F$. Addition $(a,b) \to a+b$.

$\cdot: F \times F \to F$ Product $(a,b) \to a \cdot b$.

SATISFYING THE FOLLOWING AXIOMS:

ASSOCIATIVITY: S_1: $(a+b)+c = a+(b+c)$. $\forall a, b \in F$.

P_1: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

COMMUTATIVITY: S_2: $a+b = b+a$ $\forall a, b \in F$.

P_2: $a \cdot b = b \cdot a$

IDENTITY: S_3: $\exists 0 \in F$: $a+0 = 0+a = a$ $\forall a \in F$.

P_3: $\exists 1 \in F$: $a \cdot 1 = 1 \cdot a = a$ $\forall a \in F$.

INVERSE: S_4: $\forall a \in F$. $\exists -a \in F$: $a+(-a) = (-a)+a = 0$.

P_4: $\forall a \in F$, $a \neq 0$, $\exists a^{-1} \in F$. $a \cdot a^{-1} = a^{-1} a = 1$.

DISTRIBUTIVITY D: $a(b+c) = ab+ac$ $\forall a, b, c \in F$.

Example:

1) $N = \{1, 2, 3, 4 \cdots\}$ $N \times N \xrightarrow{+} N$ (INDUCTION). $S_1 S_2$ ~~$S_3 S_4$~~, D. ⇒ NOT A FIEID.
$N \times N \xrightarrow{\cdot} N$ $P_1 P_2 P_3$ ~~P_4~~

2). $N_0 = \{0, 1, 2, 3, \cdots\}$ $N_0 \times N_0 \xrightarrow{+} N_0$ $S_1 S_2 S_3$ ~~S_4~~ D. ⇒ NOT A FIEID.
$N_0 \times N_0 \xrightarrow{\cdot} N_0$ $P_1 P_2 P_3$ ~~P_4~~

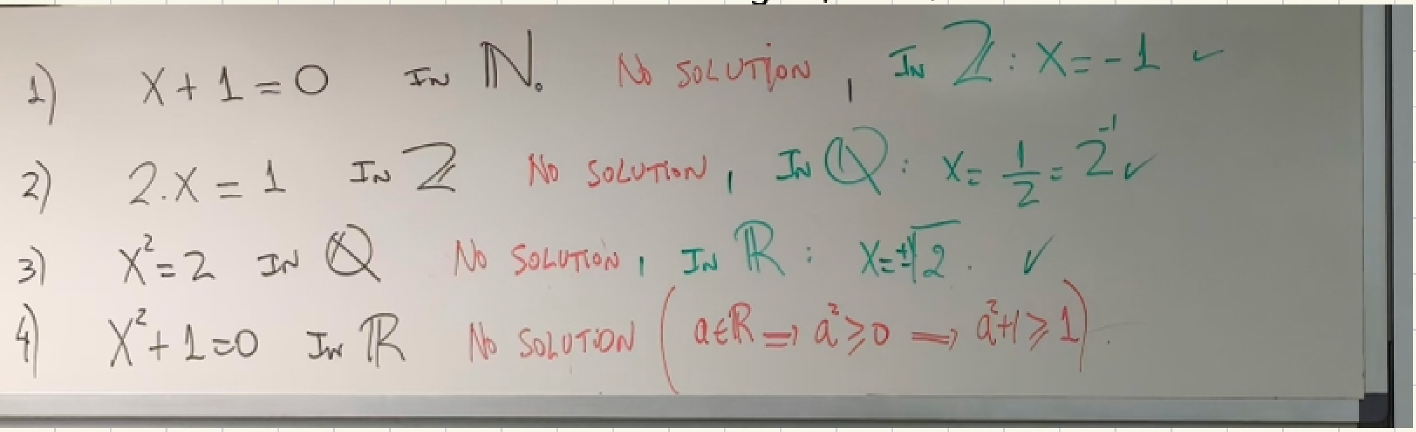
3) $Z = \{\cdots -2, -1, 0, 1, 2 \cdots\}$ $Z \times Z \xrightarrow{+} Z$ $S_1 S_2 S_3 S_4$ D. ⇒ NOT A FIELD.
$Z \times Z \xrightarrow{\cdot} Z$. $P_1 P_2 P_3$ ~~P_4~~

4) $Q = \{\frac{a}{b}, a, b \in Z, b \neq 0\}$, $Q \times Q \xrightarrow{+} Q$. $S_1 S_2 S_3 S_4$ D. ⇒ IS A FIELD.
$Q \times Q \xrightarrow{\cdot} Q$. $P_1 P_2 P_3 P_4$

5) R. $R \times R \xrightarrow{+} R$ IS A FIELD.
$R \times R \xrightarrow{\cdot} R$

$N \subseteq N_0 \subseteq Z \subseteq Q \subseteq R \subseteq C$.

PROBLEMS: Find solutions for the following equations:



1) $X + 1 = 0$ In $\mathbb{N}_0$ No SOLUTION, In $\mathbb{Z}$: $X = -1$ ✓

2) $2.X = 1$ In $\mathbb{Z}$ No SOLUTION, In $\mathbb{Q}$: $X = \frac{1}{2} = 2^{-1}$ ✓

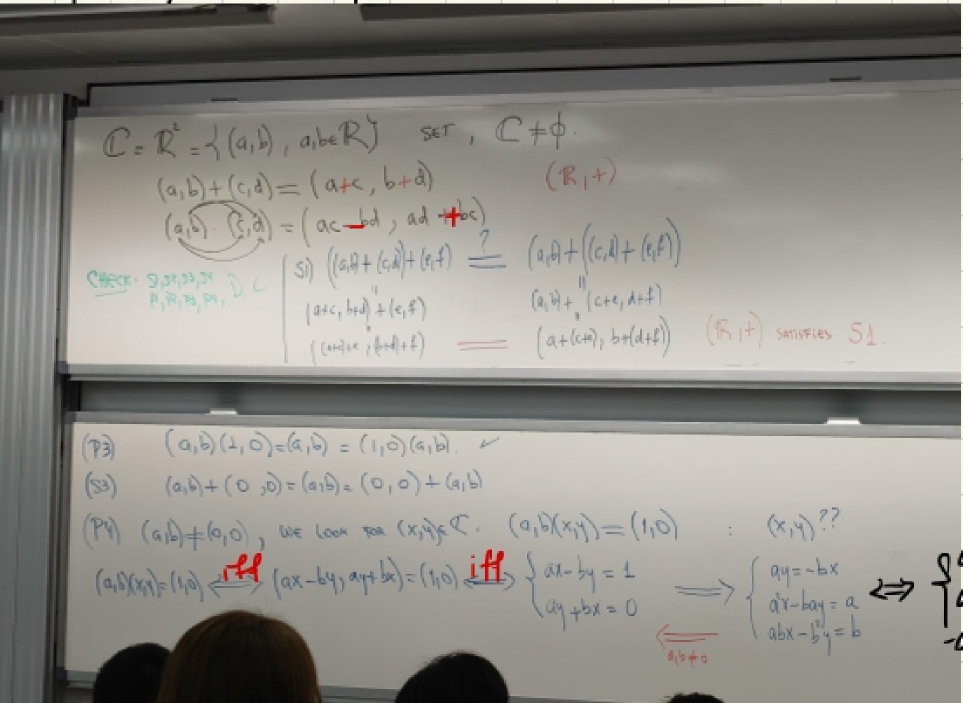
3) $X^2 = 2$ In $\mathbb{Q}$ No SOLUTION, In $\mathbb{R}$: $X = \pm\sqrt{2}$. ✓

4) $X^2 + 1 = 0$ In $\mathbb{R}$ No SOLUTION ($a \in \mathbb{R} \Rightarrow a^2 \geq 0 \Rightarrow a^2 + 1 \geq 1$).

C = Set of COMPLEX NUMBERS.

PROPERTY:

Any polynomial equation in C that has a solution in C.



$\mathbb{C} = \mathbb{R}^2 = \{(a,b), a,b \in \mathbb{R}\}$ SET, $\mathbb{C} \neq \phi$.

$(a,b) + (c,d) = (a+c, b+d)$ $(\mathbb{R}, +)$

$(a,b) \cdot (c,d) = (ac - bd, ad + bc)$

CHECK: S1, S2, S3, S4, P1, P2, P3, P4, D. ✓

S1) $((a,b) + (c,d)) + (e,f) \overset{?}{=} (a,b) + ((c,d) + (e,f))$

$(a+c, b+d) + (e,f)$ $\quad$ $(a,b) + (c+e, d+f)$

$((a+c)+e, (b+d)+f) = (a+(c+e), b+(d+f))$ $(\mathbb{R}, +)$ SATISFIES S1.

(P3) $(a,b)(1,0) = (a,b) = (1,0)(a,b)$. ✓

(S3) $(a,b) + (0,0) = (a,b) = (0,0) + (a,b)$

(P4) $(a,b) \neq (0,0)$, WE LOOK FOR $(x,y) \in \mathbb{C}$. $(a,b)(x,y) = (1,0)$: (x,y)??

$(a,b)(x,y) = (1,0) \overset{\text{iff}}{\Longleftrightarrow} (ax - by, ay + bx) = (1,0) \overset{\text{iff}}{\Longleftrightarrow} \begin{cases} ax - by = 1 \\ ay + bx = 0 \end{cases} \underset{a,b \neq 0}{\overset{\Longrightarrow}{\Longleftarrow}} \begin{cases} ay = -bx \\ a^2x - bay = a \\ abx - b^2y = b \end{cases}$

We can check if:

$(a,b) \cdot \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0)$ ✓

$a^2 + b^2 \neq 0$ since $(a,b) \neq (0,0)$.

$$\Leftrightarrow \begin{cases} ay = -bx \\ a^2x - b^2x = a \\ -a^2y - b^2y = b \end{cases} \overset{\uparrow}{\Longleftrightarrow} (x,y) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$C = R^2$ is a Field.

$R \to R^2$ INJECTIVE MAP.

$a \mapsto (a,0)$

$(a,0)+(b,0) = (a+b, 0+0) = (a+b,0)$. BEHAVE WEll WITH ADDICTION. PRODUCT.

$(a,0)\cdot(b,0) = (ab-00, a\cdot 0+0b) = (ab,0)$

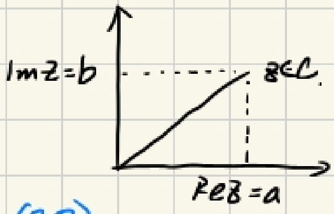
$(a,b) = (a,0)+(b,0) = (a,0)+(b,0)(0,1) = a+b(0,1) = a+bi$ Binomial Expression.

Remark:

1) $i^2 = (0,1)(0,1) = (-1,0) = -1$.

2) $(a+bi)(c+di) = (a+c)+(b+d)i$.

3) $(a+bi)\cdot(c+di) = a(c+di)+bi(c+di) = (ac-bd)+(ad+bc)i$.



DEFINITION: $z = (a,b) = a+bi \in C$.

$a = Re z$ = Real part of z. $b = Im z$ = Imaginary part of z.

$\bar{z} = (a,-b) = a-bi$ CONJUGATE of z.

$|z| = \sqrt{a^2+b^2} \in R_{\geq 0}$. MODULE of z. $\Leftarrow |z|$ = distance of (a,b) to $(0,0)$.

3.5. ClASS 2.

Def: let $z = (a,b) = a+bi \in C$.

1) MODULE: $|z| = \sqrt{a^2+b^2}$. = DISTANCE $((a,b),(0,0))$.

2) CONJUGATE: $\bar{z} = a-bi$

PROPERTIES ABOUT CONJUGATE:

1) $\bar{\bar{z}} = z$.

2) $z+\bar{z} = 2Re z$.

3) $z-\bar{z} = 2Im z\cdot i$

4) $z = \bar{z} \iff z = Re z \in R$.

5) $\overline{z+w} = \bar{z}+\bar{w}$

6) $\overline{zw} = \bar{z}\cdot\bar{w}$.

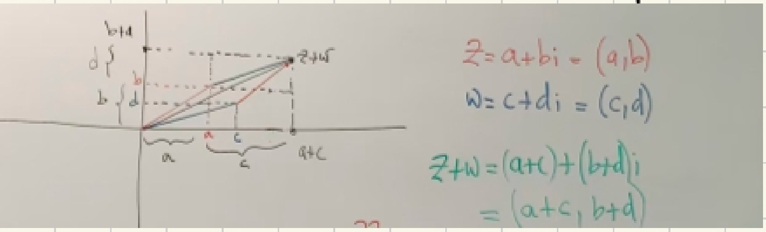
$\forall z,w \in C$.

Proof: let $z = a+bi$ $w = c+di$

5) $\overline{z+w} = \overline{a+bi+c+di} = \overline{a+c+(b+d)i} = a+c-(b+d)i \Rightarrow \overline{z+w} = \bar{z}+\bar{w}$.

$\bar{z}+\bar{w} = \overline{a+bi}+\overline{c+di} = a-bi+c-di = a+c-(b+d)i$.

GEOMETRIC INTERPRETATION of $z+w$.



$z = a+bi = (a,b)$

$w = c+di = (c,d)$

$z+w = (a+c)+(b+d)i$
$= (a+c, b+d)$

PROPERTIES OF MODULE:

1) $z\cdot\bar{z} = |z|^2$

2) $|z| = |-z| = |\bar{z}|$

3) $|z\cdot w| = |z|\cdot|w|$.

4) $\frac{|z|}{|w|} = \left|\frac{z}{w}\right|$, $w \neq 0$. $(|z+w| \neq |z|+|w|)$.

Proof:

(1) $z\cdot\bar{z}=(a+bi)(a-bi)=a^2+bai-abi-b^2i^2=a^2+b^2=(\sqrt{a^2+b^2})^2=|z|^2$.

To prove (3), we should prove the lemma: $x=y \Leftrightarrow x^2=y^2$ for $x,y\geq 0$ $x,y\in R$.

$x^2=y^2 \Leftrightarrow x^2-y^2=0 \Leftrightarrow (x+y)(x-y)=0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$. ⊠

So By the DEFINITION, $|z|=\sqrt{a^2+b^2}\in R$, and $|z|\geq 0$. $|z\cdot w|=|z|\cdot|w| \Leftrightarrow |z\cdot w|^2=(|z|\cdot|w|)^2$.

$|z\cdot w|^2=(zw)(\overline{zw})=(zw)\cdot(\bar{z}\cdot\bar{w})=(z\cdot\bar{z})(w\cdot\bar{w})=|z|^2\cdot|w|^2=(|z|\cdot|w|)^2$ ⊠

Remark:

(1) CONJUGATE of COMPLEX NUMBERS extends the absolute value of REAL NUMBERS.

$z=(a,0)=a+0i$, $|z|=\sqrt{a^2+0^2}=\sqrt{a^2}=|a|$

(2)

$z=a+bi\neq 0 \Rightarrow z^{-1}=\frac{a}{a^2+B}+\frac{-b}{a^2+b^2}i$

Since $z\cdot\bar{z}=|z|^2=a^2+b^2\in R_{\geq 0} \Rightarrow z\cdot\bar{z}\cdot\frac{1}{a^2+b^2}=1$. So $z^{-1}=\bar{z}\cdot\frac{1}{a^2+b^2}$.

So $z^{-1}=(a-bi)\cdot\frac{1}{a^2+b^2}=\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i$

PROPERTIES:

1) $Re z\leq|Re z|\leq|z|$

2) $|z+w|\leq|z|+|w|$ TRIANGLE

LAMMA: $x,y\in R$, $x,y\geq 0$ then $x\geq y \Leftrightarrow x^2\geq y^2$.

Proof: $x^2\geq y^2 \Leftrightarrow x^2-y^2\geq 0 \Leftrightarrow (x+y)(x-y)\geq 0$ since $x+y\geq 0$

So $x-y\geq 0 \Leftrightarrow x\geq y$.

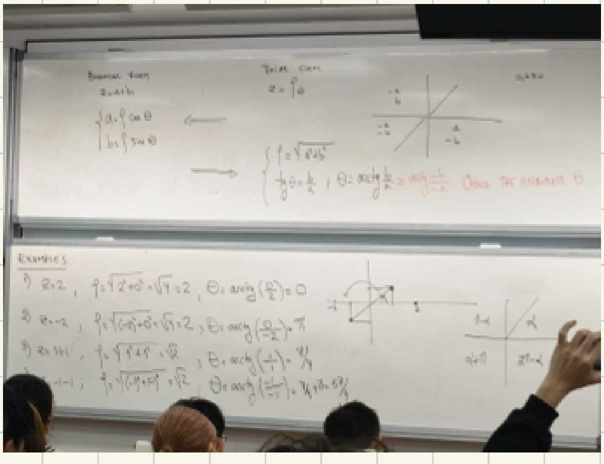
Remark In particular: $\sqrt{x}\geq\sqrt{y} \Leftrightarrow x\geq y\geq 0$.

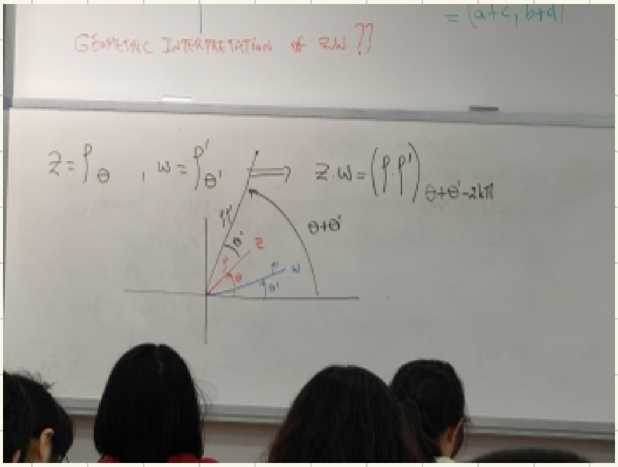
Proof:

1) $z=a+bi \Rightarrow a\leq|a|=\sqrt{a^2}\leq\sqrt{a^2+b^2}$. since $a^2\leq a^2+b^2$

$\Rightarrow Re z\leq|Re z|\leq|z|$

2) $|z+w|\leq|z|+|w| \Leftrightarrow |z+w|^2\leq(|z|+|w|)^2$.





3.10 . CLASS 3.

BINOMIAL FORM. $z = a+bi = (a,b)$.

$(a,b)+(c,d) = (a+c, b+d)$.

$(a,b)(c,d) = (ac-bd, ad+bc)$.

$a\cdot(b,c) = (a,0)\cdot(b,c) = (ab-0\cdot c, ac+0\cdot b) = (ab, ac)$.

POLAR FORM

$z = P_\theta$, $P = |z|$ $\theta = \operatorname{Arg} z$. if $z = a+bi$. $P = \sqrt{a^2+b^2}$ $\theta = \operatorname{Arg}(\frac{b}{a})$ $(\theta \in [0, 2\pi))$

$$\begin{cases} a = P\cos\theta \\ b = P\sin\theta \end{cases} \longrightarrow z = P_\theta.$$

$z = P(\cos\theta + i\sin\theta) = P\operatorname{cis}\theta = P\cdot e^{i\theta}$.

Remark: if $z = 0 \Leftrightarrow |z| = 0$ POLAR FORM FOR $z=0$ IS $P=0$. θ is not define.

POLAR FORM FOR PRODUCT.

Theorem: If $z, w \in \mathbb{C}$, $z, w \neq 0$. Then the polar form of zw is given by:

$|zw| = |z|\cdot|w|$.

$\operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w - 2k\pi$. For $k \in \mathbb{Z}$.

PROOF:

$z\cdot w = (P_z\cos(\operatorname{Arg} z) + P_z\sin(\operatorname{Arg} z)\, i)(P_w\cos(\operatorname{Arg} w) + P_w\sin(\operatorname{Arg} w)\cdot i)$.

$= P_z\cdot P_w\,(\cos(\operatorname{Arg} z + \operatorname{Arg} w) + i\sin(\operatorname{Arg} z + \operatorname{Arg} w))$

COROLLARY: If $z \in \mathbb{C}$, $n \in \mathbb{N} \Rightarrow$ the polar form of z^n is:

$|z^n| = |z|^n$.

$\operatorname{Arg}(z^n) = n\cdot\operatorname{Arg}(z) - 2k\pi$. for $k \in \mathbb{Z}$.

$\Rightarrow$ easy to prove by induction.

Example: $z = (1+i)^{1342}$.

Since $|1+i| = \sqrt{2}$. $\Rightarrow$ $1+i = \sqrt{2}\cdot e^{i\frac{\pi}{4}}$

So $|z| = |(1+i)^{1342}| = |1+i|^{1342} = (\sqrt{2})^{1342} = 2^{671}$ $\operatorname{Arg}(z) = 1342\cdot\operatorname{Arg}(1+i) + 2k\pi, k \in \mathbb{Z}$.

$\operatorname{Arg}(z) = 1342\cdot\frac{\pi}{4} + 2k\pi = \frac{(671-4k)\pi}{2} \in [0, 2\pi)$.

Now, we need to compute k: $0 \le \frac{(671-4k)}{2}\pi < 2\pi$. $\Leftrightarrow$ $0 \le 671-4k < 4$.

This is a RIMINDER of divition by 4.

$671 = 4\cdot 167 + 3$. So $\operatorname{Arg}(z) = \frac{(671-4k)\pi}{2} = \frac{3}{2}\pi$. So $z = 2^{671}e^{i\frac{3}{2}\pi} = -2^{671}i$.

ROOT.

PROBLEM: we want to find all $z \in \mathbb{C}$. that satisfy $z^n = w$. For some $w \in \mathbb{C}$ $n \in \mathbb{N}$.

REMARK: If $w=0 . \Rightarrow z^n = 0 \Leftrightarrow |z|^n = |z^n| = 0 \Leftrightarrow |z| = 0 . \Leftrightarrow z = 0$.

THEO: Let $w \in \mathbb{C}$, $w \neq 0$. $n \in \mathbb{N}$ then:

$z^n = w \Rightarrow z = \sqrt[n]{|w|}\cdot(\cos\frac{\operatorname{Arg} w + 2k\pi}{n} + i\sin\frac{\operatorname{Arg} w + 2k\pi}{2})$. $k = 0, 1, 2 \cdots n-1$

Proof: $z^n = w \Leftrightarrow |z^n| = |w|$ And $\text{Arg}(z^n) = \text{Arg}w$. $\Leftrightarrow |z|^n = |w|$ And $n\cdot\text{Arg}z - 2k\pi = \text{Arg}w \in [0,2\pi)$.

$\Leftrightarrow |z| = \sqrt[n]{|w|}$ = unique positive real n-root of the positive real number $|w|$.

And $\text{Arg}z = \frac{\text{Arg}w + 2k\pi}{n} \in [0,2\pi)$, $k \in \mathbb{Z}$.

$\Rightarrow 0 \le \frac{\text{Arg}w+2k\pi}{n} < 2\pi$. $\Leftrightarrow 0 \le \text{Arg}w + 2k\pi < 2n\pi$.

$\Rightarrow -\text{Arg}w \le 2k\pi < 2n\pi - \text{Arg}w$. since $0 \le \text{Arg}w < 2\pi$. $\Rightarrow -2\pi < -\text{Arg}w \le 0$.

$\Leftrightarrow -2\pi < -\text{Arg}w \le 2k\pi < 2n\pi - \text{Arg}w \le 2n\pi - 0$

$\Leftrightarrow -2\pi < 2k\pi < 2n\pi \Leftrightarrow -1 < k < n$. $\Leftrightarrow k = 0, 1, 2, 3, \cdots n-1$

Example: For all $z \in \mathbb{C}$. & $z^4 = 16$.

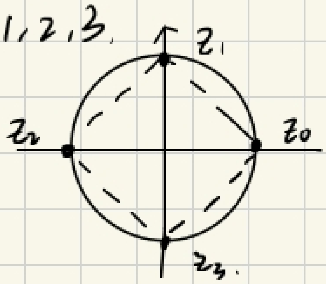
$z_k = \sqrt[4]{16}\,\text{cis}\left(\frac{0+2k\pi}{4}\right)$. $k = 0, 1, 2, 3$.

$z_0 = 2\cdot\text{cis}0 = 2$.

$z_1 = 2\cdot\text{cis}\frac{\pi}{2} = 2i$

$z_2 = 2\cdot\text{cis}\pi = -2$.

$z_3 = 2\cdot\text{cis}\frac{3\pi}{2} = -2i$.

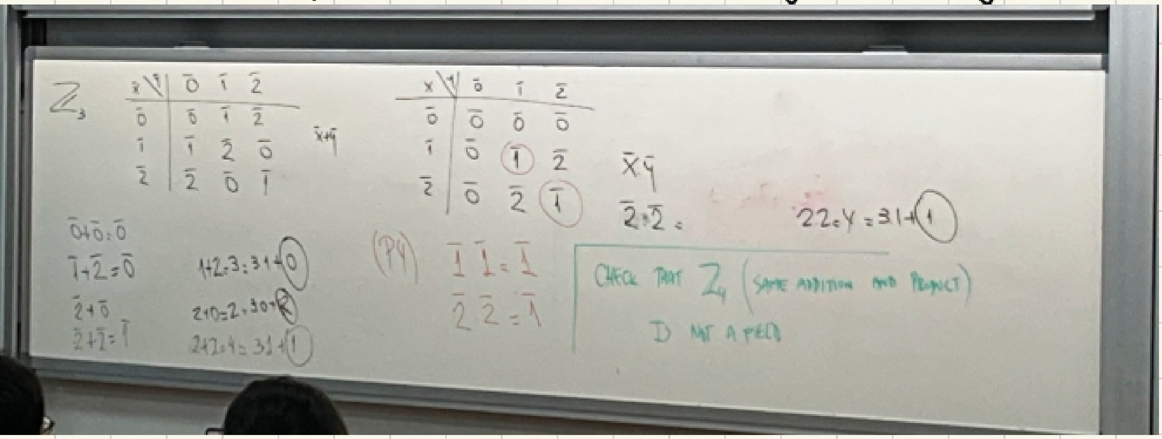


F = FIELDS.

EXAMPLES: $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

OTHER EXAMPLES: $\mathbb{Q}[i] = \{a+bi,\ a,b \in \mathbb{Q}\}$. $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2},\ a,b \in \mathbb{Q}\}$.

Finite field: $\mathbb{Z}_p = \{\bar{0}, \bar{1}, \bar{2}, \bar{3} \cdots \overline{p-1}\}$, $\bar{x} + \bar{y} = \bar{r}$. $\bar{x}\bar{y} = \bar{r}'$.



$\mathbb{Z}_3$

$x \backslash y$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

$\bar{x}+\bar{y}$

$x \backslash y$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{1}$

$\bar{x}\bar{y}$

$\bar{2}\cdot\bar{2} =$

$2\cdot2 = 4 = 3\cdot1 + 1$

$\bar{0} + \bar{0} = \bar{0}$

$\bar{1} + \bar{2} = \bar{0}$ $\quad 1+2 = 3 = 3\cdot1 + 0$

$\bar{2} + \bar{0}$ $\quad 2+0 = 2 = 3\cdot0 + 2$

$\bar{2} + \bar{2} = \bar{1}$ $\quad 2+2 = 4 = 3\cdot1 + 1$

(P4) $\bar{1}\,\bar{1} = \bar{1}$

$\bar{2}\,\bar{2} = \bar{1}$

CHECK THAT $\mathbb{Z}_4$ (SAME ADDITION AND PRODUCT) IS NOT A FIELD

POLYNOMIALS ON ONE VARIABLE X WITH COEFFICIENTS ON A FIELD F.

$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $a_i \in F$, $n \in \mathbb{N}$. x is an Indeterminate.

$= (a_0, a_1, a_2, a_3 \ldots a_n, 0, 0, \ldots)$

DEFINITIONS $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n = g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots + b_mx^m$.

$\Leftrightarrow a_k = b_k$. $\forall k \geq 0$.

Example: (1) $f(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n = 0 = 0 + 0x + 0x^2 + \cdots + 0x^n$.

$(a_0, a_1, a_2, a_3 \cdots a_n, 0, 0 \ldots)$ $\quad (0, 0, 0, 0, 0 \ldots)$

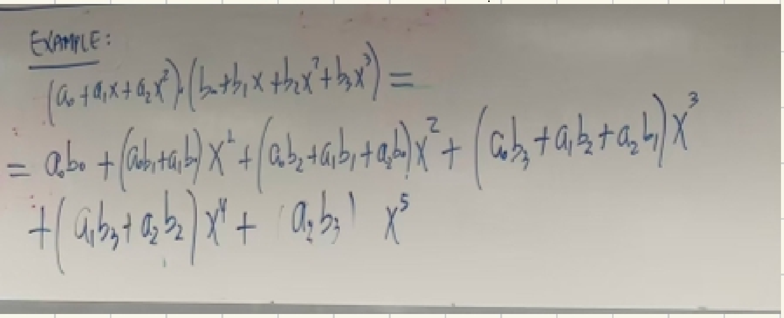
$\Leftrightarrow a_0 = a_1 = a_2 \cdots = a_n = 0$.

(2) $f(x) = a_0x + a_1x^2 + a_2x^3 + \cdots + a_nx^n = 0 \Leftrightarrow a_k = 0\ \forall k \geq 0$.

(3) $F[x]$ = set of All POLYNOMIALS IN ONE VARIABLE WITH COEFFICENTS In F.
(4) We can define an ADDITION And a PRODUCT In $F(x)$.

$F(x) \times F(x) \xrightarrow{+} F(x)$.
$\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i$

$F(x) \times F(x) \xrightarrow{\times} F(x)$
$\sum a_i x^i \times \sum b_i x^i = \sum (\sum_{i+j=k} a_i b_j) x^k$.



EXAMPLE:

$(a_0 + a_1 x + a_2 x^2) \cdot (b_0 + b_1 x + b_2 x^2 + b_3 x^3) =$
$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x^1 + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1) x^3$
$+ (a_1 b_3 + a_2 b_2) x^4 + (a_2 b_3) x^5$

$(F(x), +, \cdot)$ satisfies S_1, S_2, S_3, S_4. D.
$P_1, P_2, P_3,$ ~~P_4~~ (check that these is no $f(x)$ st $f(x) \cdot x = 1$)
$\Downarrow$
NOT A FIELD.
REMARK: Compare with $(\mathbb{Z}, +, \cdot)$ Without P4, we called it RINGS.

DEF: Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in F(x)$ $f(x) \neq 0$.
1) $\deg f(x) = \max\{k, a_k \neq 0\}$.
2) If $n = \deg f(x)$, a_n = LEADING COEFFICIENT.
3) If $n = \deg f(x)$, $a_n = 1$ $f(x)$ is called MONIC.

PROPOSITION: If $f(x), g(x) \in F[x]$, $f(x) \cdot g(x) \neq 0$, then:
1) $f(x) \cdot g(x) \neq 0$.
2) $\deg(f(x) \cdot g(x)) = \deg f(x) + \deg g(x)$.
3) If $f(x) + g(x) \neq 0$ then $\deg(f(x) + g(x)) \leq \max\{\deg f(x), \deg(g(x))\}$.

3.12. CLASS 4.

In $\mathbb{Z}$, Any $n \in \mathbb{Z}$, $n \neq 0, 1, -1$ can be written in a UNIQUE way as (± 1) times a product of positive PRIME NUMBER.

$(\mathbb{Z}, +, \cdot)$ RING $\leftarrow$ positive prime numbers.
$(F[x], +, \cdot)$ RING $\leftarrow$ Monic IRREDUCIBLE POLYNOMIALS.

Def: A polynomial $f(x) \in F[x]$ is called IRREDUCIBLE if $f(x) \neq 0$, $\deg(f(x)) \neq 0$. And if $f(x) = g(x) \cdot h(x)$, $g, h \in F[x]$. $\Rightarrow \deg(g(x)) = 0$ or $\deg(h(x)) = 0$.

Remark: $f(x)$ is IRREDUCIBLE $\Leftrightarrow$ $f(x) \neq 0$, and it cannot be written as the product of two non-constant polynomials.

Example: $x^2 - 1 = (x+1)(x-1)$ NOT IRREDUCIBLE
$x - a = g(x) \cdot h(x) \Rightarrow \deg(x-a) = 1 = \deg(g(x)) + \deg(h(x)) \Rightarrow g(x) = 0$ or $h(x) = 0$.

So $g(x)$ or $h(x)$ is constant , $x-a$ is IRREDUCIBLE.

$x^2+1=(x-i)(x+i)$ NOT IRREDUCIBLE in $C[x]$

x^2+1 is IRREDUCIBLE In $Q[x]$ and $R[x]$.

$x^2-3=(x-\sqrt{3})(x+\sqrt{3})$ NOT IRREDUCIBLE in $R[x]$ or in $C[x]$.

x^2-3 is IRREDUCIBLE in $Q[x]$.

THEO: Any non-constant polynomial in $F[x]$ can be written in a UNIQUE As a constant times of Monic IRREDUCIBLE POLYNOMIALS.

(USE INDUCTION TO PROVE BY Deg $f(x)$).

Remark: IT IS NOT EASY TO DETERMINE IF A POLYNOMIALS $f(x)$ IS IRREDUCIBLE OR NOT.
AND IT IS NOT EASY TO FACTORIZE A POLYNOMIAL INTO A PRODUCT.

THE EASY IDEA is FIND ROOTS.

DIVISION ALGORITHM IN $F[x]$.

for any $f(x), g(x) \in F[x]$, $g(x) \neq 0$, these exist UNIQUE $q(x), r(x) \in F[x]$ st
$f(x)=g(x)\cdot q(x)+r(x)$, $r(x)=0$ or $\deg(r)<\deg(g)$.

Proof: $X=\{f(x)-g(x)q(x),\ q,g\in F[x]\}$. If $0\in X \Rightarrow 0=f(x)-g(x)\cdot q(x)$, $r(x)=0$ ✓.
if $0\notin X$, $\phi\neq \deg(X)=\{\deg(f(x)-q(x)\cdot g(x)),\ q,g\in F[x]\}\subseteq N_0$. WOP: $\exists s$ first element in $\deg X$.
$s=\deg(f(x)-g(x)q(x))$ Suppose $\deg(r)\geq\deg(g)$ ---- Contradiction.

Definition: let $f(x)\in F[x]$, $a\in F$, say that a is a root of $f(x)$ if $f(a)=0$.

Example: 2 is the root of x^3-3x-2 since $2^3-3\cdot2-2=0$.
1 is not root of x^3-3x-2 since $1^3-3\cdot1-2=-4\neq0$.

REMAINDER THEOREM

let $f(x)\in F[x]$, $a\in F$. the remainder of the division of $f(x)$ by $x-a$ is $f(a)$

Proof:

By the DIVISION ALGORITHM. $\exists!\, q(x), r(x)\in F[x]$: $f(x)=(x-a)\cdot q(x)+r(x)$ with
$r(x)=0$ or $\deg(r)<\deg(x-a)=1 \Rightarrow \deg(r)=0 \Rightarrow r(x)=c \Rightarrow$ constant.
$f(x)=(x-a)q(x)+c$. $f(a)=0\cdot q(a)+c \Rightarrow c=f(a) \Rightarrow f(x)=(x-a)q(x)+f(a)$. ☒

COROLLARY: let $f(x)\in F[x]$, $f(x)\neq0$. THEN a is a root of $f(x) \Leftrightarrow f(a)=0 \Leftrightarrow f(x)=(x-a)q$

3.7. CLASS 5.

VECTOR SPACES.

Def: let F be a field, V be a set. then V is an F-vector space if there are two operations:

ADDITION OF VECTORS: $+: V \times V \longrightarrow V$

$(v, w) \rightarrow v + w$.

PRODUCT BY SCALARS: $\cdot: F \times V \longrightarrow V$.

$(\lambda, v) \longrightarrow \lambda v$.

satisfying:

S1) **ASSOCIATIVITY**: $(v+w)+u = v+(w+u)$. $\forall v, w, u \in V$.

S2) **COMMUTATIVITY**: $v+w = w+v$, $\forall v, w \in V$.

S3) **IDENTITY**: $\exists 0 \in V$: $v + 0_v = v = 0_v + v$, $\forall v \in V$

S4) **INVERSE**: $\forall v \in V$. $\exists -v \in V$, $v + (-v) = 0_v = (-v) + v$.

M1) $(\lambda + u)\cdot v = \lambda \cdot v + u \cdot v$, $\forall \lambda, u \in F$. $v \in V$

M2) $(\lambda \cdot u)\cdot v = \lambda(u \cdot v)$, $\forall \lambda, u \in F$, $v \in V$.

M3) $\lambda \cdot (v+w) = \lambda \cdot v + \lambda \cdot w$ $\forall \lambda \in F$. $v, w \in V$.

M4) $1 \cdot v = v$ $\forall v \in V$.

V = SET OF VECTORS.

F = SET OF SCALARS.

Question: If (M4) say that $1 \cdot v = v$, why we say nothing about $0 \cdot v$?

Since

$$\underset{\text{from } F}{0} \cdot v = \underset{\text{from } V}{0}.$$

Properties: Let V be an F-vector space, $v, w, u \in V$. $\lambda \in F$.

1) $v + u = w + u \Rightarrow v = w$.

$u + v = u + w \Rightarrow v = w$.

2) $0_F \cdot v = 0_v$; $\lambda \cdot 0_v = 0_v$.

3) $\lambda \cdot v = 0_v \Rightarrow \lambda = 0_F$ or $v = 0_v$.

4) $-v = (-1_F) \cdot v$.

Proof:

1) $v + u = w + u \xRightarrow{S_4} (v+u) + (-u) = (w+u) + (-u)$ since $-u \in V \xRightarrow{S_1} v + (u-u) = w + (u-u)$. $\xRightarrow{S_4} v + 0_v = w + 0_v \xRightarrow{S_3} v = w$. $u + v = u + w \Rightarrow v = w$ use the same way. ■

2) $0_F \cdot v + 0_v \overset{S_3}{=} 0_F \cdot v \overset{S_3}{=} (0_F + 0_F) \cdot v \overset{M_1}{=} 0_F \cdot v + 0_F \cdot v \Rightarrow 0_v = 0_F \cdot v$. ■

$\lambda \cdot 0_v + 0_v \overset{S_3}{=} \lambda \cdot 0_v \overset{S_3}{=} \lambda \cdot (0_v + 0_v) \overset{M_3}{=} \lambda 0_v + \lambda 0_v \Rightarrow \lambda \cdot 0_v = 0_v$.

3) $\lambda \cdot v = 0_v$ we want to see that $\lambda = 0_F$ or $v = 0_v$. If $\lambda = 0_F$. we done, If $\lambda \neq 0_F$. By P4. $\exists \lambda^{-1} \in F$. $0_v \overset{12}{=} \lambda^{-1} 0_v = \lambda^{-1}(\lambda \cdot v) = (\lambda^{-1} \cdot \lambda) \cdot v = 1_F v = v$.

$\Rightarrow 0_v = v$.

4) $-v = (-1_F)v \iff v + (-1)\cdot v = 0_v$ Since $v + (-1)v \overset{M_4}{=} 1\cdot v + (-1)v \overset{M_1}{=} (1+(-1))v = 0_F \cdot v \overset{(2)}{=} 0_v$.

MORE PROPETIES:

1) 0_v is unique.

2) For any $v \in V$, $-v$ is unique.

Proof:

(1) If $0, 0'$ are IDENTITIES In V satisfying (S_3), then:

$$0' \underset{0 \text{ identity}}{=} 0 + 0' \underset{0' \text{ identity.}}{=} 0.$$ we have done.

(2) If v_1, v_2 are INVERSES OF v. then:

$$v_1 = v_1 + 0_v = v_1 + (v + v_2) \overset{S_1}{=} (v_1 + v) + v_2 = 0_v + v_2 = v_2$$

$\Rightarrow v_1 = v_2$ ⊠

Remark: $(S_3) \Rightarrow V \neq \phi$ since $0_v \in V$.

Example:

1) $V = \{*\}$ a set with only one element in F-vector space.

$V \times V \to V$. $F \times V \to V$.

$(*, *) \to * + *$. $(\lambda, *) \to \lambda \cdot *$.

and satisfies $S_1 \to S_4$, $M_1 \to M_4$.

2) $V = F$ is an F-vector space.

$(F, +, \cdot)$ field $\quad (\underset{\text{vector}}{F},\ \underset{\text{ADDITION OF VECTOR.}}{F \times F \xrightarrow{+} F},\ \underset{\text{PRODUCT OF VECTORS}}{F \times F \xrightarrow{\cdot} F})$ VECTOR SPACE.

$+ = +$ $\quad \cdot = \cdot$

$S_1 = S_1$ $\quad S_2 = S_2$ $\quad (S_3): 0_v = 0_F$, $\quad S_4 = S_4$ $\quad M_1 = D$ $\quad M_2 = P_1$ $\quad M_3 = D$ $\quad M_4 = P_3$.

R is an R-VECTORS SPACE.

C is an C-VECTORS SPACE.

3) $(F, +, \cdot)$ FIELD, $V = F \times F = \{(x,y) : x, y \in F\}$.

$$(x,y) + (x', y') = (x + x', y + y')$$
$$\lambda(x,y) = (\lambda \cdot x, \lambda \cdot y)$$

S_2. $(x,y) + (x' \cdot y') = (x + x', y + y') = (x' + x, y' + y) = (x'y') + (x, y)$.

S_3. $0_v = (0,0)$ since $(x,y) + (0,0) = (x+0, y+0) = (x,y)$.

S_4. $-(x,y) = (-x, -y)$.

4) $(F, +, \cdot)$ FIELD, $V = F^n = \overset{\nearrow \text{ for } n \text{ times}}{F \times F \times F \times F \times \cdots F} = \{(x_1, x_2, \ldots x_n) : x_1, x_2, x_3 \cdots x_n \in F\}$

$(x_1, x_2 \ldots x_n) + (x_1', x_2', x_3' \cdots x_n') = (x_1 + x_1'; x_2 + x_2', \ldots x_n + x_n')$ $\quad \lambda(x_1, x_2, x_3 \cdots x_n) = (\lambda x_1, \lambda x_2, \cdots \lambda x_n)$.

5) (F, +, ·) FIELD. $V = F[x]$ = SET OF POLYNOMIALS.

$F[x]\times F[x] \xrightarrow{+} F[x]$ Addition of polynomials.

$F\times F[x] \xrightarrow{\cdot} F[x]$ Production by a constant polynomial.

6) (F, +, ·) FIELD. $V = M_{n\times m}(F) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}),\ a_{ij}\in F. \right\}$

$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

$\lambda(a_{ij}) = (\lambda\cdot a_{ij})$

7) Let V be an F-vector space, let $X \neq \emptyset$ set : $V\times V \xrightarrow{+} V$. $F\times V \xrightarrow{\cdot} V$.

$W = V^X = \{f: X\to V \text{ functions}\}$; vectors. $f+g: X\to V: (f+g)(x) = f(x)+g(x)$.

is an F-vector space. $\lambda\cdot f: X\to V: (\lambda f)(x) = \lambda\cdot f(x)$.

8) $C^2 = \{(a+bi, c+di), a,b,c\in R\}$ = VECTORS.

$(a+bi, c+di) + (a'+b'i, c'+d'i) = (a+a'+(b+b')i, c+c'+(d+d')i)$

$(x+yi)(a+bi, c+di) = ((x+yi)(a+bi), (x+yi)(c+di)) \Longrightarrow C^2$ is a C-vector space.

Remark: C^2 also an R-vector space $C^2\times C^2 \xrightarrow{+} C^2$ then $R\times C^2 \longrightarrow C^2$.

SUBSPACE.

Definition.

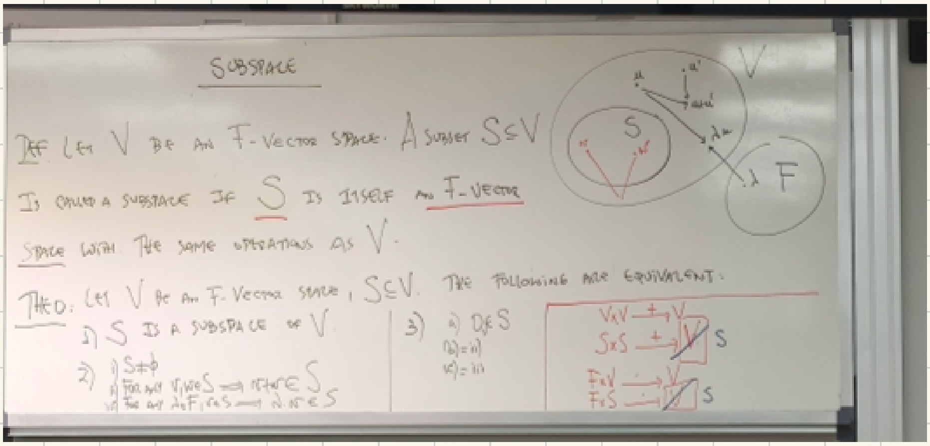
let V be an F-vector space. A subset $S\subseteq V$ is called a subspace if S is an F-vector space with the same operations as V.

THEO: let V be an F-vector space $S\subseteq V$. The following are equivalent:

1) S is a subspace of V.

2) (i) $S\neq\emptyset$ (ii) For any $v, w\in S \Rightarrow v+w\in S$. (iii) For any $\lambda\in F, v\in S \Rightarrow \lambda\cdot v\in S$.

3) (a) $0_V\in S$ (b) = (ii) (c) = (iii).



SUBSPACE

DEF: LET V BE AN F-VECTOR SPACE. A SUBSET $S\subseteq V$ IS CALLED A SUBSPACE IF S IS ITSELF AN F-VECTOR SPACE WITH THE SAME OPERATIONS AS V.

THEO: LET V BE AN F-VECTOR SPACE, $S\subseteq V$. THE FOLLOWING ARE EQUIVALENT.

1) S IS A SUBSPACE OF V.

2) i) $S\neq\emptyset$
ii) FOR ANY $v,w\in S \to v+w\in S$
iii) FOR ANY $\lambda\in F, v\in S \to \lambda\cdot v\in S$

3) a) $0_V\in S$
b) = ii)
c) = iii)

$V\times V \xrightarrow{+} V$
$S\times S \xrightarrow{+} \cancel{V}\ S$
$F\times V \xrightarrow{\cdot} V$
$F\times S \xrightarrow{\cdot} \cancel{V}\ S$

3.19. CIASS 6.

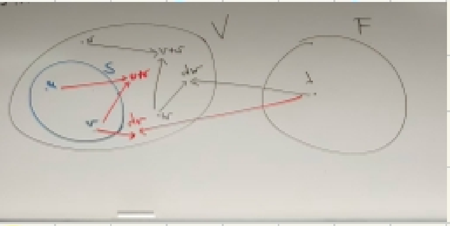
F-vector space = $(F,+,\cdot)$ field, $(V,+)$ + $S_1 S_2 S_3 S_4$. $F\times V \xrightarrow{\cdot} V$ + $M_1 M_2 M_3 M_4$.

SUBSPACE: $S \subseteq V$. S is an F-vector space with the same operations as V.

Question: how do we check if S is an F-vector space?

Since $V\times V \xrightarrow{+} V$ $\quad F\times V \xrightarrow{\cdot} V$. $\Rightarrow$ Restrict it into S.

$S\times S \xrightarrow{+} V$. $\quad F\times S \xrightarrow{\cdot} V$. if $S\times S \xrightarrow{+} S$. $F\times S \xrightarrow{\cdot} S$. ✓.



THEO: Let V be an F-vector space, let $S \subseteq V$. The following are equal:

a) S is a subspace (= S is an F-vector space with the operations from V).

b) i) $S \neq \phi$. ii) If $v, w \in S \Rightarrow v+w \in S$. iii) if $\lambda \in F$. $v \in S \Rightarrow \lambda v \in S$.

c) i) $0 \in S$. ii) $v+w \in S$, iii) $\lambda \in F$, $v \in S$. $\lambda v \in S$.

What we need to prove: $a) \Rightarrow b)$
$\qquad \nwarrow c) \swarrow$

Proof $a) \Rightarrow b)$.

i). We know that any vector space is not-empty. Then S is not empty.

ii. iii) If S is an F-vector space with the same operation in V. Then restrictions: $S\times S \xrightarrow{+} S$, $F\times S \xrightarrow{\cdot} S$. Then if $v.w \in S \Rightarrow v+w \in S$. $\lambda \in F$. $v \in S$, $\lambda v \in S$. ■

$b) \Rightarrow c)$.

We just need to prove i).

$S \neq \phi$. $\exists v \in S$ by iii). $0_F \in F$ $v \in S$. $\Rightarrow 0_F \cdot v = 0_v \in S$. ■

$c) \Rightarrow a)$.

By c). we know that S has addition and product. $(F,+,\cdot)$. $S\times S \xrightarrow{+} S$. $F\times S \xrightarrow{\cdot} S$.

Now we check the axiom:

S_1) $v+w+u = v+(w+u)$ $\forall v,w,u \in S$ This holds since it holds $\forall v,w,u \in V$.

S_2) we know $v+w = w+v$ $\forall v,w \in V$. So in particular, $v+w = w+v$ $\forall v,w \in S$.

S_3) = i). $0_v \in S$.

S_4) $v \in S$, we know that $-v = (-1)\cdot v$ by iii). $-1 \in F$. $v \in S \Rightarrow (-1)\cdot v \in S$.

M_1) $\lambda\cdot(v+w) = \lambda\cdot v + \lambda\cdot w$ $\forall \lambda \in F$, $v,w \in S$.

M_2) $(\lambda+\mu)\cdot v = \lambda\cdot v + \mu\cdot v$ $\forall \lambda,\mu \in F$. $v \in S$.

M_3) $(\lambda\cdot\mu)v = \lambda(\mu\cdot v)$ $\forall \lambda,\mu \in F$. $v \in S$.

M_4) $1\cdot v = v$ $\forall v \in S$.

} This hold, since they are in V.

Example:

1) Let V be an F-vector space.

$S=\{0\}$ is a subspace $\begin{cases} 0\in S. \\ 0,0\in S \quad 0+0=0\in S. \\ \lambda\in F,\ 0\in S.\ \lambda\cdot 0=0\in S. \end{cases}$

$S=V$ is a subspace $\begin{cases} 0\in V. \\ v+w\in V. \\ \lambda\in F.\ v\in V.\ \lambda v\in V \end{cases}$

2) $V=R^3=\{(x,y,z),\ x,y,z\in R\}$. is an R-vector space.

$T=\{(x,y,1),\ x,y\in R\}\subseteq V$ is not a subspace since $0_{R^3}=(0,0,0)\notin T$.

and $(x,y,1)+(x',y',1)=(x+x',y+y',2)\notin T$.

$S=\{(x,y,0):x,y\in R\}$ is a subspace.

$(0,0,0)\in S$ ✓. $(x,y,0)+(x',y',0)=(x+x',y+y',0)\in S$ ✓.

$\lambda(x,y,0)=(\lambda x,\lambda y,0)\in S$. ✓

3) $V=R^3$ $S=\{(x,y,z):ax+by+z=0,\ x,y,z\in R\}\subseteq R^3$.

Since $(0,0,0)\in S$. $a\cdot 0+b\cdot 0+c\cdot 0=0$ ✓.

$(x,y,z)+(x',y',z')=(x+x',y+y',z+z')$ $a(x+x')+b(y+y')+c(z+z')=0$. ✓

Let $\lambda\in R$. $\lambda(x,y,z)=(\lambda x,\lambda y,\lambda z)$. since $ax+by+cz=0\Rightarrow \lambda ax+\lambda\cdot by+\lambda c\cdot z=0$. ✓.

4) $\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{mn}x_n=b_m \end{cases} \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{mn} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

$A \cdot x = b$.

Let $T=\{(x_1,x_2\cdots x_n):Ax=b\}\subseteq F^n$. ← F-vector space.

$(0,0,0\cdots 0)\in T \Leftrightarrow b_1=b_2=\cdots b_n=0$.

T is not a subspace if $b_1 b_2\cdots b_n\neq 0$. If $(b_1\cdots b_n)=(0,0,\cdots 0)\Rightarrow T=\{(x_1,x_2\cdots x_n):A\cdot x=0\}$ is a subspace.

$(0,0,0\cdots 0)\in T$.

$x=(x_1,x_2\cdots x_n)$, $x'=(x_1',x_2',x_3'\cdots x_n')\in T\Rightarrow Ax=0.\ Ax'=0.\Rightarrow A(x+x')=Ax+Ax'=0.\Rightarrow x+x'\in T$.

$x=(x_1,x_2,\cdots x_n)\in T$, $\lambda\in F$. $\Rightarrow A\cdot(\lambda\cdot x)=A(\lambda\cdot Id)\cdot x=\lambda(Ax)=(\lambda\cdot Id)\cdot 0=0.\Rightarrow \lambda\cdot x\in T$.

5) C is a C-vector space, $R\subseteq C$. R is not a subspace of C vector space.

i) $0\in R$, ii) $x,y\in R.\Rightarrow(x+0i)+(y+0i)=x+y=(x+y)+0i\in C$. ✓.

iii) $\lambda\in C$. $x\in R.\Rightarrow\lambda x\in R.\Rightarrow$ Not true since if $\lambda=i$. ✗

In otherwise. C is an R-vectorspace.

$R\subseteq C$, R is a subspace.

i) $0=0+0i\in R$. ii) $x,y\in R,\Rightarrow x+y=x+0i+y+0i=(x+y)+0i\in R$.

iii) $\lambda\in R$, $x\in R.\Rightarrow\lambda\cdot x=\lambda(x+0i)=\lambda x+0i\in R$.

6) $R^R=\{f:R\to \underline{R}$ functions$\}$. → R-vector space $\Rightarrow R^R$ is an R-vector space.

We have already checked $S_1S_2S_3S_4M_1M_2M_3M_4$ ✓.

Let $C(R)=\{f:R\to R$ continuous$\}$ is a subspace of R^R.

let a function $O(x)$. $O: R\to R$, $O(x)=0$ is continuous. then $O(x) \in C(R)$

let $f, g \in C(R) \Rightarrow f+g \in C(R)$. $\lambda \in R$. $f \in C(R) \Rightarrow \lambda f \in C(R)$.

$\lim(f+g) = \lim f + \lim g$. $\lim(\lambda f) = \lambda \cdot \lim f$.

THEO: If S_1, S_2 are subspace of a F-vector space of V. $\Rightarrow S_1 \cap S_2$ is a subspace of V.

2). If $\{S_i, i \in T\}$ ... ... $\Rightarrow \bigcap_{i\in I} S_i$ ... ...

Proof:

1). $0 \in S_1$, $0 \in S_2$. $\Rightarrow 0 \in S_1 \cap S_2$ ✓.

If $v, w \in S \cap S_2 \Rightarrow \begin{cases} v, w \in S_1 \Rightarrow v+w \in S_1 \\ v, w \in S_2 \Rightarrow v+w \in S_2 \end{cases} \Rightarrow v+w \in S_1 \cap S_2$.

If $\lambda \in F$, $v \in S_1 \cap S_2 \Rightarrow \begin{cases} \lambda \in F, v \in S_1 \Rightarrow \lambda \cdot v \in S_1 \\ \lambda \in F, v \in S_2 \Rightarrow \lambda \cdot v \in S_2. \end{cases} \Rightarrow \lambda \cdot v \in S_1 \cap S_2$.

Example:

$S_1 = \{(x,y,0) : x, y \in R\} \subseteq R^3$. $S_2 = \{(x,0,z), x, z \in R\} \subseteq R^3$ both are subspace.

$S_1 \cap S_2 = \{(x,0,0) : x \in R\}$ subspace ✓.

$S_1 \cup S_2 = \{(x,y,0) : x,y \in R\} \cup \{(x,0,z) : x,z \in R\}$ is NOT a subspace.

Since $(1,1,0) + (1,0,1) = (2,1,1) \notin S_1 \cup S_2$.

Def:

1) Let $S_1 . S_2$ be subspace of an F-vector space V. Then S_1+S_2 is the SMAllES subspace of V. containing $S_1 \cup S_2$. with respect to inclu.

Proof: S_1+S_2 is a subspace containing $S_1 \cup S_2$, $S_1+S_2 \subseteq R^3$.

$(x,y,0) \in S_1 \subseteq S_1+S_2 \subseteq R^3$. $(0,0,z) \in S_2 \subseteq S_1+S_2 \subseteq R^3$.

$(x,y,0) + (0,0,z) \in S_1+S_2$, S_1+S_2 is a subspace $\Rightarrow (x,y,z) = (x,y,0) + (0,0,z) \in S_1+S_2$.

$\Rightarrow R^3 \subseteq S_1+S_2 \subseteq R^3$. $\Rightarrow S_1+S_2 = R^3$.

Def:

2) Let $S_1, S_2 \cdots S_n$ be subspace of V. Then $S_1+S_2+\cdots+S_n$ is the smallest subspace of V. Containing $S_1 \cup S_2 \cup S_3 \cdots \cup S_n$.

THEO:

1) $S_1+S_2 = \{v_1+v_2, v_1 \in S_1, v_2 \in S_2.\}$

2) $S_1+S_2+\cdots+S_n = \{v_1+v_2+\cdots+v_n, v_1 \in S_1, v_2 \in S_2 \cdots v_n \in S_n\}$.

PROOF: WE HAVE TO PROVE THAT S_1+S_2 IS THE $\overset{(c)}{\underline{\text{SMALLEST}}}$ $\overset{(a)}{\underline{\text{SUBSPACE}}}$. $\overset{(b)}{\underline{\text{CONTAINING}}}$ S_1 and S_2.

(a) i) $0 = \overset{\in S_1}{0} + \overset{\in S_2}{0} \in S_1+S_2$. since S_1, S_2 are subspace.

ii) v_1+v_2, w_1+w_2, $v_1, w_1 \in S_1$. $v_2, w_2 \in S_2 \Rightarrow (v_1+v_2)+(w_1+w_2) = \overset{\in S_1}{(v_1+w_1)} + \overset{\in S_2}{(v_2+w_2)} \in S_1+S_2$.

iii) $\lambda \in F$. v_1+v_2 $v_1 \in S_1$ $v_2 \in S_2 \Rightarrow \lambda(v_1+v_2) = \lambda v_1 + \lambda v_2 \in S_1+S_2$.

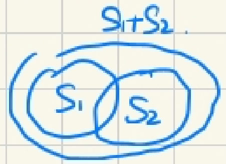
(b) $S_1 \subseteq S_1+S_2$ since $v \in S_1 \Rightarrow v = \overset{\in S_1}{v} + \overset{\in S_2}{0} \in S_1+S_2$.

$S_2 \subseteq S_1+S_2$ since $v \in S_2 \Rightarrow v = \overset{\in S_2}{v} + \overset{\in S_1}{0} \in S_1+S_2$.

(c) We have to prove that S_1+S_2 is the SMALLEST one. let U be a subspace containing $S_1 \cup S_2$.

We have to prove that $S_1+S_2 \subseteq U$.

$v_1+v_2 \in S_1+S_2 \Rightarrow \begin{cases} v_1 \in S_1 \subseteq U \\ v_2 \in S_2 \subseteq U \end{cases} \Leftrightarrow v_1, v_2 \in U$. U is a subspace, $\Rightarrow v_1+v_2 \in U$.



Remark: S_1+S_2 is unique.

U_1 is the smallest subspace containing $S_1 \cup S_2$. $\Rightarrow U_1 \subseteq U_2$, $U_2 \subseteq U_1$

U_2 is the smallest subspace containing $S_1 \cup S_2$

Example:

$\mathbb{R}^3 = S_1+S_2 = \{(x,y,0) : x,y \in \mathbb{R}\} + \{(x,0,z), x,z \in \mathbb{R}\}$.

$(x,y,z) = (x,y,0) + (0,0,z) \in S_1+S_2$. Not unique.

$= (0,y,0) + (x,0,z) \in S_1+S_2$.

$\mathbb{R}^3 = S_1+S_2 = \{(x,y,0) : x,y \in \mathbb{R}\} + \{(0,0,z), z \in \mathbb{R}\}$.

$(x,y,z) = (a,b,0) + (0,0,c) \Leftrightarrow (x,y,z) = (a,b,c)$

$= (x,y,0) + (0,0,z)$ In a unique way.

3.24. CLASS 7

Def: the sum of two subspaces S_1, S_2 of an F-vector space V is the $\overset{(iii)}{\underline{\text{SMALLEST}}}$ $\overset{(i)}{\underline{\text{SUBSPACE}}}$ of V that contains $\underline{S_1 \cup S_2}$. (ii)

THEO: $S_1+S_2 = \{v_1+v_2, v_1 \in S_1, v_2 \in S_2\}$.

Proof: Let $X = \{v_1+v_2, v_1 \in S_1, v_2 \in S_2\}$. We want to prove that $X = S_1+S_2$.

Then we have to check.

i) X is a subspace of V.

ii) $S_1 \cup S_2 \subseteq X$

iii) If U is a subspace of V. $S_1 \cup S_2 \subseteq U \Rightarrow X \subseteq U$

$\Big\} \Rightarrow X = S_1+S_2$.

Example:

$\mathbb{R}^3 = \{(x,y,0), x,y \in \mathbb{R}\} + \{(0,y,z) : y,z \in \mathbb{R}\} \overset{\text{THEO}}{=} \{(x,y,0)+(0,y',z'), x,y,y',z' \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

For any $(x,y,z) \in \mathbb{R}^3$, we have $(x,y,z) = (x,y,0) + (0,0,z) \in S_1+S_2$.

$= (x,0,0) + (0,y,z) \in S_1+S_2$.

$\Rightarrow$ The way in which we write (x,y,z) as v_1+v_2, $v_1 \in S_1$, $v_2 \in S_2$ is not unique.

$$\mathbb{R}^3 = \underbrace{\{(x,y,0), x,y \in \mathbb{R}\}}^{S_1} \oplus \overbrace{\{(0,0,z) : z \in \mathbb{R}\}}^{S_2}$$

$(x,y,z) = (x,y,0) + (0,0,z)$

$= (a,b,0) + (0,0,c) \Leftrightarrow \begin{cases} a = x \\ b = y \\ c = z \end{cases}$.

Def: 1) A sum of two subsequences, S_1+S_2, is called DIRECT SUM if any vector in S_1+S_2 can be written in a unique way as $v_1+v_2 \in S_1+S_2$ $v_1 \in S_1$, $v_2 \in S_2$.

That is, $w = v_1+v_2 = v_1'+v_2'$ $v_1, v_1' \in S_1$ $v_2, v_2' \in S_2 \Rightarrow \begin{cases} v_1 = v_1' \\ v_2 = v_2' \end{cases}$.

2). A sum of n subspace $S_1+S_2+\dots S_n$ is DIRECT SUM if $w = v_1+v_2+v_3+\dots+v_n = v_1'+v_2'+v_3'+\dots+v_n' \in S_1+S_2+\dots+S_n$.

for $v_1, v_1' \in S_1$, $v_2, v_2' \in S_2 \dots v_n, v_n' \in S_n$

$\Rightarrow v_1 = v_1' \quad v_2 = v_2' \cdots \quad v_n = v_n'$

$\Rightarrow S_1 \oplus S_2 \oplus S_3 \cdots \oplus S_n$

Example:

$R^3 = S_1 + S_2 \Rightarrow$ is not direct.

$= S_1 \oplus S_3 = \{(x,y,0), x,y \in R\} \oplus \{(0,0,z), z \in R\}$. ①

$= T_1 \oplus T_2 \oplus S_3 = \{(x,0,0), x \in R\} \oplus \{(0,y,0), y \in R\} \oplus \{(0,0,z), z \in R\}$. ②

If we have any (x,y,z). let's do ① way:

$(x,y,z) = (a,b,0) + (0,0,c)$. $\Rightarrow a = x,\ b = y,\ c = z$. then for $\forall (x,y,z)$ we have only one solution.

② way:

$(x,y,z) = (a,0,0) + (0,b,0) + (0,0,c) \Rightarrow a = x.\ b = y.\ c = z$ " . . . ".

Now, let's assume $(x,y,z) = (a,0,0) + (0,b,b) + (c,0,c)$. $\Leftrightarrow \begin{cases} x = a+c \\ y = b \\ z = b+c \end{cases} \Leftrightarrow \begin{cases} b = y \\ c = z-y. \\ a = x-z+y. \end{cases}$

PROBLEMS:

1) How do we know that $V = S_1 + S_2$?

If $S_1 + S_2 \subseteq V$ and any $w \in V$ we can be written as $v_1 + v_2$, $v_1 \in S_1$. $v_2 \in S_2 \Rightarrow w \in S_1 + S_2$

2) How do we know that $V = S_1 \oplus S_2$?

For any $v \in V$ check that if it can be written as $v_1 + v_2 \in S_1 + S_2$ st $v_1 \in S_1$, $v_2 \in S_2$ in a unique way.

We have a theo.

THEOREM:

1) $S_1 \oplus S_2$ is a direct sum $\Leftrightarrow 0 = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2 \Rightarrow v_1 = 0$, $v_2 = 0$.

2) $S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is a direct sum $\Leftrightarrow 0 = v_1 + v_2 + \cdots + v_n$, $v_i \in S_i$, $i = 1,2 \ldots n$. $\Rightarrow v_1 = v_2 = v_3 = \cdots = v_n = 0$.

Proof:

1) $\Rightarrow$) We know that any $w \in S_1 + S_2$ can be written in a unique way as $w = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2$ In particular:

We know that $0 = \overset{\in S_1}{0} + \overset{\in S_2}{0}$ in a unique way.

Then $0 = \overset{\in S_1}{v_1} + \overset{\in S_2}{v_2} \Rightarrow v_1 = 0$. $v_2 = 0$.

$\Leftarrow$) Assume $w = v_1 + v_2 = v_1' + v_2'$ $v_1, v_1' \in S_1$, $v_2, v_2' \in S_2$. We want to prove that $v_1 = v_1'$ $v_2 = v_2'$.

Since $0 = w - w = v_1 + v_2 - (v_1' + v_2') = \overset{\in S_1}{v_1 - v_1'} + \overset{\in S_2}{v_2 - v_2'}$ By hypothesis $v_1 - v_1' = 0$ $v_2 - v_2' = 0$. $\Rightarrow v_1 = v_1'$ $v_2 = v_2'$. ∎

Corollary: $S_1 + S_2 = S_1 \oplus S_2$ is a direct sum $\Leftrightarrow S_1 \cap S_2 = \{0\}$.

Proof: We know that $S_1 + S_2 = S_1 \oplus S_2 \Leftrightarrow 0 = v_1 + v_2$, $v_1 \in S_1$ $v_2 \in S_2 \Rightarrow v_1 = 0$, $v_2 = 0$.

let's see that $(0 = \overset{\in S_1}{v_1} + \overset{\in S_2}{v_2} \Rightarrow v_1 = 0 = v_2) \Leftrightarrow S_1 \cap S_2 = \{0\}$.

$\Rightarrow$) $v \in S_1 \cap S_2 \Rightarrow -v \in S_1 \cap S_2$. Since $S_1 \cap S_2$ is a subspace and $-v = (-1) \cdot v$.

$0 = \overset{\in S_1 \cap S_2}{v} + \overset{\in S_1 \cap S_2}{(-v)} \Rightarrow \begin{cases} v = 0 \\ -v = 0 \end{cases} \Rightarrow \{0\} \subseteq S_1 \cap S_2 \subseteq \{0\} \Rightarrow S_1 \cap S_2 = \{0\}$ (we know that $\{0\} \subseteq S_1 \cap S_2$ trivially).

$\Leftarrow$) let $0 = v_1 + v_2$ $v_1 \in S_1$ $v_2 \in S_2 \Rightarrow \overset{\in S_1}{v_1} = \overset{\in S_2}{-v_2}$. $\in S_1 \cap S_2 = \{0\}$.

$\Rightarrow \begin{cases} v_1 = 0 \\ -v_2 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_2 = 0 \end{cases} \Rightarrow S_1 \oplus S_2$ ∎

$0 = v_1 + v_2$. for $v_1 \in S_1$. $v_2 \in S_2$.

$v_1 = -v_2$. $\Rightarrow v_1 \in S_2 \Rightarrow v_1 \in S_1 \cap S_2$.

$v_2 \in S_1 \cap S_2$. $\Rightarrow v_1 = v_2 = 0$. $\Rightarrow S_1 \oplus S_2$.

Example: let $S_1 = \{(x,0,x), x \in R\}$. $S_2 = \{(x,0,-x), x \in R\}$.

$S_1 \oplus S_2 = \{(a,0,a)+(b,0,-b),\ a,b \in \mathbb{R}\} = \{(a+b,0,a-b),\ a,b\in\mathbb{R}\}$.

let's take any $(x,0,y)$ for $x,y\in\mathbb{R}$. we have: $x=a+b.\ y=a-b. \Leftrightarrow \begin{cases} a=\frac{x+y}{2} \\ b=\frac{x-y}{2} \end{cases}$

Question = What is $S_1 \cap S_2$?

$(x,y,z)\in S_1\cap S_2 \Leftrightarrow \begin{matrix}(x,y,z)\in S_1 \\ (x,y,z)\in S_2\end{matrix} \Leftrightarrow \begin{cases} y=0 \text{ and } x=z \\ y=0 \text{ and } x=-z \end{cases} \Leftrightarrow x=y=z=0 \Rightarrow S_1\cap S_2=\{(0,0,0)\}$

Remark: The corollary is not true for $n\geqslant 3$

$S_1+S_2+S_3+\cdots S_n = S_1\oplus S_2\oplus S_3\cdots\oplus S_n \not\Leftrightarrow S_1\cap S_2=\{0\},\ S_2\cap S_3=\{0\}\ \cdot\cdot S_{n-1}\cap S_n=\{0\}$.

BUT $S_1+S_2+S_3 = S_1\oplus S_2\oplus S_3 \Leftrightarrow S_1\cap(S_2+S_3)=\{0\}\quad S_2\cap(S_1+S_3)=\{0\}$.

We will see in Algebra B.

LINEAR COMBINATION

definition:

let V be an F-vector space $v_1,v_2\cdots v_n,\ w\in V$.

We say that w is a Linear Combination (LC) of $v_1,v_2\cdots v_n$ if $\exists \lambda_1,\lambda_2\ldots\lambda_n\in F$ st $w=\lambda_1 v_1+\lambda_2 v_2+\cdots+\lambda_n\cdot v_n$.

Example:

1) Check that $(1,0,1)$ is a L.C of $(2,1,3)$ $(1,0,0)$ and $(0,1,1)$.

$(1,0,1)=\lambda_1(2,1,3)+\lambda_2(1,0,0)+\lambda_3(0,1,1) \Leftrightarrow \begin{cases}1=2\lambda_1+\lambda_2\\0=\lambda_1+\lambda_3\\1=3\lambda_1+\lambda_3\end{cases} \Leftrightarrow \begin{cases}\lambda_1=-\lambda_3\\3\lambda_1-\lambda_1=1\\\lambda_2=2\lambda_1-1\end{cases} \Leftrightarrow \begin{cases}\lambda_1=-\lambda_3\\\lambda_1=\frac{1}{2}\\\lambda_2=0\end{cases}$

$=\frac{1}{2}(2,1,3)+(-\frac{1}{2})(0,1,1)$.

Def: $S_1+S_2+\cdots+S_n$ = Sum of subspaces = SMALLEST SUBSPACE of V. contain $S_1\cup S_2\cdots\cup S_n$

Def: $S_1\oplus S_2$ = Direct sum of " ".

Def: L.C. = sum of scalars times vectors.

Definition: let V be an F-vector space, $v_1\cdots v_n\in V$.

SPAN$(v_1,v_2\ldots v_n)$ = SPAN$(\{v_1,v_2\ldots v_n\})$. is the smallest (with respect to function). subspace of V containing the set $\{v_1,v_2,\cdots v_n\}$.

Remark:

let $U=\text{span}(v_1,v_2\cdots v_n)$. U is a subspace and $v_i\in U$ for any $i=1,2,\cdots n$.

$v_i\in U \Leftrightarrow S_i=\{\lambda\cdot v_i,\ \lambda\in F\}\subseteq U$.

$\Rightarrow)$ $v_i\in U\Rightarrow \lambda v_i\in U$. since U is a subspace

$\Leftarrow)$ $v_i=1\cdot v_i\in U$

THEO: Span$(v_1,v_2\cdots v_n)$ = smallest subspace of U containing $S_1\cup S_2\cup S_3\cdots\cup S_n$. $S_i=\{\lambda\cdot v_i,\ \lambda\in F\}$.

$=S_1+S_2+\cdots+S_n$.

$=\{\lambda_1 v_1,\lambda_2 v_2,\lambda_3 v_3,\cdots\lambda_n v_n,\ \lambda_i\in F\}$ = set of L.C. of $v_1,v_2\cdots v_n$.

Definition

We say that the set $\{v_1,v_2,\cdots v_n\}$ spans the F-vector space V if

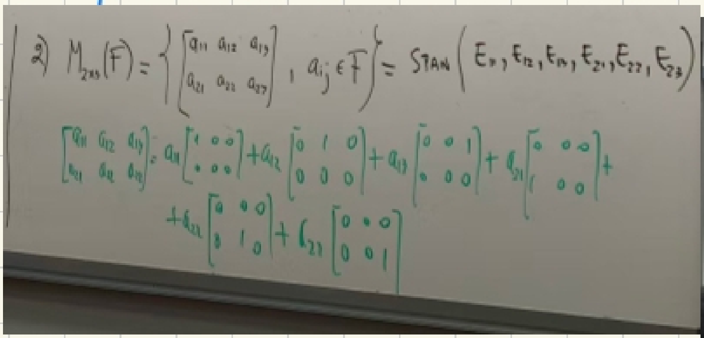
$V=\text{span}(v_1,v_2\cdots v_n)$ = set of L.C. of $v_1,v_2\ldots v_n$ $=\{\lambda v_1,\lambda\in F\}+\{\lambda v_2,\lambda\in F\}\cdots+\{\lambda v_n,\lambda\in F\}$.

Example:

1) $F^2=\{(x,y): x,y\in F\} = span(\{(1,0),(0,1)\})$. $(x,y)=x\cdot(1,0)+y\cdot(0,1)$.

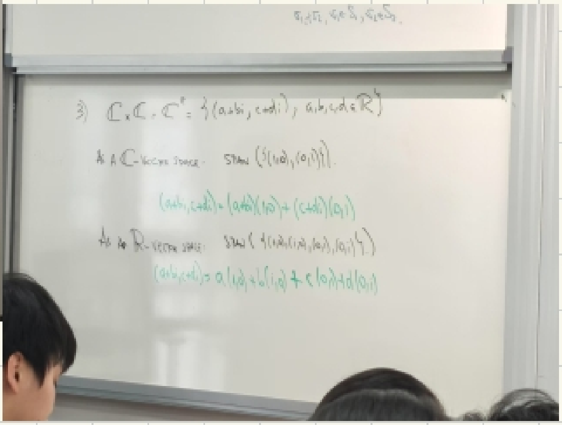
$F^n = span(\{P_1,P_2,\cdots P_n\})$ $P_1=(1,0,0,\cdots 0)$. $P_2=(0,1,0,0\cdots 0)$ $\cdots$ $P_n=(0,0,\cdots 0,1)$.

2)



2) $M_{2\times3}(F)=\left\{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}, a_{ij}\in F\right\} = \text{SPAN}(E_{11},E_{12},E_{13},E_{21},E_{22},E_{23})$

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}=a_{11}\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}+a_{12}\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}+a_{13}\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}+a_{21}\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}+$$
$$+a_{22}\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}+a_{23}\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}$$



3) $\mathbb{C}\times\mathbb{C}=\mathbb{C}^2=\{(a+bi,c+di), a,b,c,d\in\mathbb{R}\}$

As a $\mathbb{C}$-vector space: $\text{SPAN}(\{(1,0),(0,1)\})$.

$(a+bi,c+di)=(a+bi)(1,0)+(c+di)(0,1)$

As an $\mathbb{R}$-vector space: $\text{SPAN}(\{(1,0),(i,0),(0,1),(0,i)\})$

$(a+bi,c+di)=a(1,0)+b(i,0)+c(0,1)+d(0,i)$

3.26. ClASS 8.

Vector space: $(F,+,\cdot)$ field, $(V,+)$, $F\times V\rightarrow V$. } + Axioms.

F-subspaces = subsets of F-vector space which are F-vector space.

Intersetion of subspaces is a subspace

Union is not always a subspace

Sum on subspace: $S_1+S_2+\cdots+S_n \overset{\text{Def}}{=}$ SMAllEST SUBSPACE OF V CONTAINING $S_1\cup S_2\cup S_3\cdots\cup S_n$.

$\overset{\text{THEO}}{=} \{v_1+v_2+\cdots+v_n, v_i\in S_i, i=1,2,\cdots n\}$.

In particular: $S_i=\{\lambda v_i, \lambda\in F\}$ (Check that S_i is a subspace of V)

$S_1+S_2+\cdots+S_n=\{\lambda_1v_1+\lambda_2v_2+\cdots+\lambda_nv_n, \lambda_1\cdots\lambda_n\in F\}$.

$\overset{\text{Def}}{=}$ set of LC of $v_1\cdots v_n$.

$\overset{\text{Def}}{=}$ SMAllEST SUBSET OF V CONTAINING $S_1\cup S_2\cup\cdots\cup S_n$.

$\overset{\text{Proposition}}{=}$ " " " " $\{v_1,\cdots v_n\}$.

$\overset{\text{Def}}{=}$ SPAN$(v_1\cdots v_n)$.

Proposition: U is a vector space: $v\in U \Leftrightarrow \{\lambda v_i, \lambda\in F\}\subseteq U$.

Example:

1) C-vector space $C^2 \Rightarrow C^2=\text{SPAN}((1,0),(0,1))$.

$(a+bi, c+di) = (a+bi)\cdot(1,0)+(c+di)\cdot(0,1)$.

$\{(1,0),(0,1)\}$ does not span the R-vector space C^2: $(i, 1+i)\neq u_1(1,0)+u_2(0,1)$, $u_1,u_2\in R$.

2) R-vector space $C^2\Rightarrow C^2=span\{(1,0),(i,0),(0,1),(0,i)\}$.

$(a+bi, c+di)=a(1,0)+b(i,0)+c(0,1)+d(0,i)$. $a,b,c,d\in R$.

3) Let $S=\{(x+y,x,2y): x,y\in R\}\subseteq R^3$. Find a spanning set of S.

$(x+y,x,2y)=(x,x,0)+(y,0,2y).=x(1,1,0)+y(1,0,2)\Rightarrow S=span\{(1,1,0),(1,0,2)\}$

4). $\text{Span}((1,2,0),(0,1,1)) \overset{\text{DEF}}{=}$ SMALLEST SUBSPACE OF R^3 CONTAINING $\{(1,2,0),(0,1,1)\}$.

$= \{a(1,2,0)+b(0,1,1) : a,b\in R\}$.

$= \{(a, 2a+b, b) . a,b\in R\} \not\ni (2,3,10)$.

$= \{(x,y,z) : y = 2x+z , x,z\in R\}. = \{(x, y, y-2x) . y, x\in R\}$.

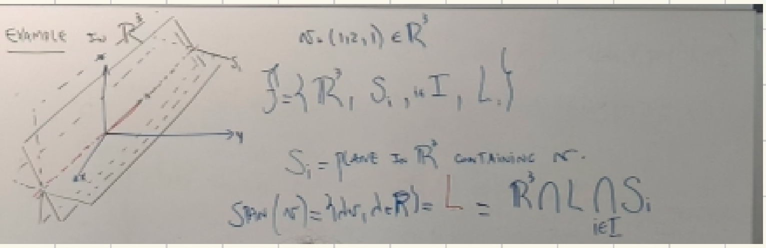
$= \text{span}((1,0,2),(0,1,1))$.

THEO: let V be an F-vector space , $X\subseteq V$ a subset.

Consider : $f = \{S\subseteq V , S$ is a subspace of $V , X\subseteq S\}$ Then $\text{span}(X) = \bigcap_{S\in f} S$.

Proof: $f\neq\phi$ since $V\in f$ $\text{span}(x)=\bigcap_{S\in f} S \Leftrightarrow \bigcap_{S\in f} S$ is a subspace of V.

$\begin{cases} x\subseteq \bigcap_{S\in f} S. & \text{since } X\subseteq S \ \forall S\in f \\ \text{SMALLEST}. & \bigcap_{S\in f} S\subseteq S \ \forall S\in f \end{cases}$



Remark:

1) $S_1+S_2+\cdots+S_n = \{v_1+v_2+\cdots+v_n , v_i\in S_i\} = \{\lambda_1 v_1+\lambda_2 v_2+\cdots+\lambda_n v_n , v_i\in S_i\}$.

$= \text{Span}(X)$. $X = S_1\cup S_2\cup\cdots\cup S_n$.

We can apply the theorem for $X=S_1\cup S_2\cup\cdots\cup S_n$.

2) $\text{SPAN}\{v_1\cdots v_n\}$, $X=\{v_1 , v_2 , v_3 \cdots v_n\}$.

We can apply the theorem for this X.

FINITE DIMENSIONAL VECTOR SPACES

Def: An F-vector space V is finite dimensional if $\exists X\subseteq V$. X finite , st $V=\text{span}(X)$.

That is $X=\{v_1\cdots v_n\}$, $V=\text{span}(v_1 v_2 \cdots v_n) = \{\lambda_1 v_1+\lambda_2 v_2+\cdots+\lambda_n v_n , \lambda_1,\lambda_2\cdots\lambda_n\in F\}$.

V is Infinite dimensional if it is not finite dimensional.

Notation : V has finite dimension $\dim_F V < \infty$.

V has infinite dimension $\dim_F V = \infty$.

Example :

1) $\dim_F F^n < \infty$ since $(x_1 x_2 \cdots x_n) = x_1\overset{e_1}{(1,0,\cdots 0)} + x_2\overset{e_2}{(0,1,0,0\cdots 0)} + \cdots + x_n\overset{e_n}{(0,0,\cdots 0,1)}$.

2) $\dim_F M_{m\times n}(F) < \infty$ $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn}\end{bmatrix} = a_{11}E_{11}+a_{12}E_{12}+\cdots+a_{mn}E_{mn}$. $(E_{ij})_{ks} = \begin{cases} 1 & (i,j)=(k,s) \\ 0 & (k,s)\neq(i,j).\end{cases}$

$M_{m\times n}(F) = \text{span}(E_{11}, E_{12}, \cdots E_{1n}, \cdots E_{m1} \cdots E_{mn})$.

3) $S=\text{span}((1,2,0),(0,1,1)) = \{(x, 2x+z, z) , x,z\in R\}$, $\dim_R S < \infty$.

4) $\dim_F F[x] = \infty$

Assume $\dim_F F[x] < \infty$, $\exists X=\{p_1(x), p_2(x) \cdots p_s(x)\}\subseteq F[x]$. st $F[x] = \text{span}(X) = \text{span}(p_1(x), p_2(x), \cdots p_s(x)) = \{\lambda_1 p_1(x)+\cdots+\lambda_s p_s(x), \lambda_i\in$

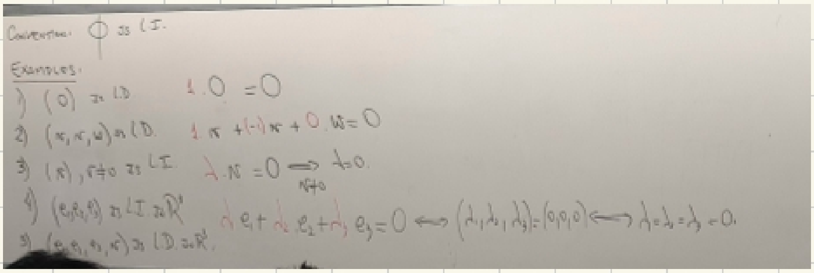
If $m = \max\{\deg p_i(x), p_i(x), i=1,2,\cdots s\}$. $\Rightarrow x^{m+1}\notin \text{span}(p_1 p_2\cdots p_s) = F[x]$. Contradiction.

Def: A sequence of vectors $(v_1 v_2 \cdots v_n)$ in an F-vector space. V is called Linearly Dependent. LD.
if $\exists \lambda_1 \lambda_2 \cdots \lambda_n \in F$, $(\lambda_1 \lambda_2 \cdots \lambda_n) \neq (0,0,0,\cdots 0)$ st $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$.

Def: A sequence is Linearly Independent LI. if is not LD.

An Infinite sequence of vectors is called LD if it contains a finite subsequence is LD

LI is in the same property.



Convention: $\emptyset$ is L.I.

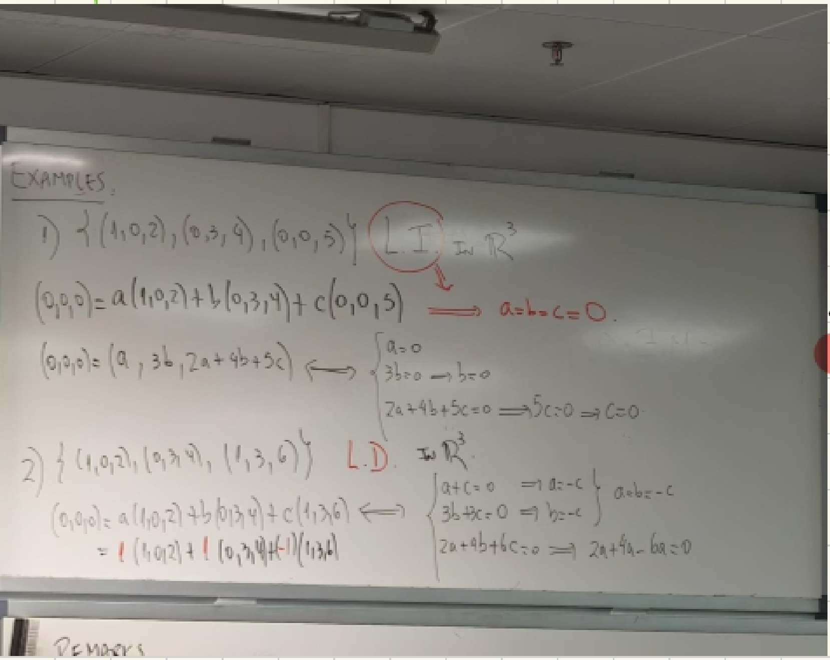
Examples:

1) (0) is LD $1.0=0$
2) (v,v,w) is LD. $1.v+(-1)v+0.w=0$
3) (v), $v\neq0$ is LI. $\lambda.v=0 \underset{v\neq0}{\Longrightarrow} \lambda=0$.
4) (e_1,e_2,e_3) is LI in $\mathbb{R}^3$ $\lambda_1 e_1+\lambda_2 e_2+\lambda_3 e_3=0 \Longleftrightarrow (\lambda_1,\lambda_2,\lambda_3)=(0,0,0) \Longleftrightarrow \lambda_1=\lambda_2=\lambda_3=0$.
5) (e_1,e_2,e_3,v) is LD in $\mathbb{R}^3$.

3.31. CLASS 9.

Remark: We always write 0 as a Trival LC of $v_1 v_2 \ldots v_n$ $0 = 0 \cdot v_1 + 0 v_2 + \cdots + 0 \cdot v_n$.
$(v_1 \cdot v_2, \ldots v_n)$ is L.D $\Rightarrow$ 0 can be written as a L.C. of $v_1 v_2 \ldots v_n$ in a non-trival way.

Example.



EXAMPLES.

1) $\{(1,0,2),(0,3,4),(0,0,5)\}$ L.I. in $\mathbb{R}^3$

$(0,0,0)=a(1,0,2)+b(0,3,4)+c(0,0,5) \Longrightarrow a=b=c=0$.

$(0,0,0)=(a,3b,2a+4b+5c) \Longleftrightarrow \begin{cases} a=0 \\ 3b=0 \rightarrow b=0 \\ 2a+4b+5c=0 \Longrightarrow 5c=0 \Rightarrow c=0 \end{cases}$

2) $\{(1,0,2),(0,3,4),(1,3,6)\}$ L.D. in $\mathbb{R}^3$.

$(0,0,0)=a(1,0,2)+b(0,3,4)+c(1,3,6) \Longleftrightarrow \begin{cases} a+c=0 \Rightarrow a=-c \\ 3b+3c=0 \Rightarrow b=-c \\ 2a+4b+6c=0 \Longrightarrow 2a+4a-6a=0 \end{cases}$ $a=b=-c$

$=1(1,0,2)+1(0,3,4)+(-1)(1,3,6)$

REMARKS

Remark:

1) $\emptyset$ is LI.
2) $\{v, v\in V, v\neq0\}$ L.I.
3) $\{v, v, v_1, v_2 \cdots v_n\}$ LD. since
$v+(-1)\cdot v + 0 \cdot v_1 + 0 v_2 + \cdots 0 v_n = 0$.
4) $(0, v_2, v_3, \ldots)$ LD. $0 = 1.0 + 0.v_2 \cdots$
5) (v, w) LI $\underline{\Leftrightarrow}$ $v \neq w \neq 0$. and $w \neq \lambda \cdot v$, $\lambda \in F$.
$\Leftrightarrow v, w \neq 0$, $w \notin span(v)$

let's prove (5).

$\Leftarrow$) By contradiction: assume (v, w) are LD. $\Rightarrow \exists a, b$ st $0 = a \cdot v + b w$
if $b=0 \Rightarrow 0 = a \cdot v$, $v \neq 0 \Rightarrow a=0 \Rightarrow (a,b)=(0,0)$ Contradiction
if $b\neq0 \Rightarrow w = -\frac{a}{b} v = \lambda \cdot v$, $\lambda = -\frac{a}{b}$ Contradiction since $w \neq \lambda \cdot v$. $w \notin span(v)$.

$\Rightarrow$) Assume (v, w) L.I. if $v=0 \Rightarrow (v,w)=(0,w)$ LD. If $w=0$ is LD. if $w=\lambda v$.
$\Rightarrow (v, w)$ is LD. Since $0 = \lambda \cdot v + (-1) \cdot w$. $\Rightarrow$ it is LD.
Therefore, the all cases are Contradiction.

EXAMPLE: INFINITE L.I. SET.

$\{1, x, x^2, x^3, \cdots x^n, x^{n+1} \ldots\}$ in $F[x]$ F is a field. is LI.

We have check that any finite subset is LI.

$0 = a_1 x^{i_1} + a_2 x^{i_2} + a_3 x^{i_3} + \cdots + a_s x^{i_s} \Leftrightarrow a_1 = a_2 = \cdots = a_s$. $i_j \neq i_k$ $j \neq k$.

since the 0 polynomial has all finite coefficients.

EXAMPLE:

$\{(1,0,1), (2,0,2)\}$ is LD. $(0,0,0) = -2(1,0,1) + (2,0,2)$.

$\Rightarrow \{(1,0,1), (2,0,2), (1,5,10), \ldots\}$ is LD.

PROBLEM: What is the connetion between L.D sets and spanning set?

Proposition: let V be an F-vector space, $\{v_1, v_2, \ldots v_n\} \subseteq V$. Then:

The list $(v_1, v_2, \ldots v_n)$ is L.D. $\Leftrightarrow \exists j$, $v_j \in span(v_1, v_2 \ldots v_{j-1})$. for $1 \leq j \leq n$.

In this case: $span(v_1, v_2, \cdots v_n) = span(v_1, v_2 \cdots v_{j-1}, v_{j+1}, \cdots v_n)$.

Proof:

$\Rightarrow$) Assume $(v_1, v_2, \cdots v_n)$ is L.D. $\Rightarrow$ $0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ $\lambda_i \in F$. st $\{k, \lambda_k \neq 0\}$ is not empty and finite. let $j = \max\{k : \lambda_k \neq 0\}$, Then:

$$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1} + \lambda_j v_j. \Rightarrow v_j = \left(\frac{-\lambda_1}{\lambda_j}\right) v_1 + \left(\frac{-\lambda_2}{\lambda_j}\right) v_2 + \left(\frac{-\lambda_3}{\lambda_j}\right) v_3 + \cdots + \left(\frac{-\lambda_{j-1}}{\lambda_j}\right) v_{j-1}.$$

$v_i \in Span(v_1, v_2 \ldots v_{j-1})$.

$\Leftarrow$). if $v_j \in span(v_1, v_2, \cdots v_{j-1}) \Rightarrow v_j = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1}$, $\lambda_1 \cdots \lambda_{j-1} \in F$.

$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1} + (-1) \cdot v_j \Rightarrow \{v_1, v_2, \cdots v_j, v_{j+1} \cdots v_n\}$ is L.D. Q

In this case, $v_j = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1}$.

$a_1 v_1 + a_2 v_2 + \cdots + a_j v_j + \cdots + a_n v_n = a_1 v_1 + a_2 v_2 + \cdots + a_{j-1} v_{j-1} + a_j(\lambda_1 v_1 + \cdots + \lambda_{j-1} v_{j-1}) + a_{j+1} v_{j+1} + \cdots + a_n v_n$

$= (a_1 + a_j \lambda_1) v_1 + (a_2 + a_j \lambda_2) v_2 + \cdots + (a_{j-1} + a_j \lambda_{j-1}) v_{j-1} + a_{j+1} v_{j+1} + \cdots + a_n v_n$

$\Rightarrow span(v_1, v_2 \cdots v_j \cdots v_n) = Span(v_1, v_2 \cdots v_{j-1}, v_j, \cdots v_n)$.

Example:

$S = span((1,0,0), (2,0,0), (0,1,0), (0,3,0), (0,0,1)) \subseteq R^3$.

$= \{a(1,0,0) + b(2,0,0) + c(0,1,0) + d(0,3,0) + e(0,0,1), a, b, c, d, e \in R\}$.

$j=1$ $(1,0,0) \notin span(\phi)$ = SMALLEST SUBSPACE CONTAINING $\phi = \{(0,0,0)\}$. $\Rightarrow j \neq 1$.

$j=2$. $(2,0,0) \in span(1,0,0)$, since $(2,0,0) = 2 \cdot (1,0,0)$. ✓.

$\Rightarrow S = span((1,0,0), (0,1,0), (0,3,0), (0,0,1))$.

To the same way, consider $j=3,4$. $\Rightarrow$ $S = span((1,0,0), (0,1,0), (0,0,1))$.

Remark:

$j=1 \Leftrightarrow v_1 \in Span(\phi) = \{0\} \Leftrightarrow v_1 = 0$.

$j=2 \Leftrightarrow v_2 \in span(v_1) \Leftrightarrow \{v_1, v_2\}$ L.D.

$j=3 \Leftrightarrow v_3 \in span(v_1, v_2) \Leftrightarrow \{v_1, v_2, v_3\}$ L.D.

$v_3 = \lambda_1 v_1 + \lambda_2 v_2$.

Proposition: Let V be an F-vector space, $\{v_1, v_2, v_3 \cdots v_m\}$ is a LI set. $\{w_1, w_2, \cdots w_n\}$ spanning set, that is $V = span(w_1, w_2, \dots w_n)$. Then $m \le n$.

THE CARDINALITY OF ANY LI SET $\le$ CARDINALITY OF ANY SPANNING SET.

Example:

$R^3 = \{(x,y), x,y \in R\} = \{(x,y) = x(1,0) + y(0,1)\} = span((1,0),(0,1))$. $n=2$

If $\{(a,b), (c,d), (e,f)\}$ L.D $m = 3 \nleq 2$. $\Rightarrow$ It is not a subspace of R^2.

Now, we need to prove the proposition.

let $V = span(w_1, w_2, w_3 \dots w_n)$, $B_0 = \{w_1, w_2, \dots w_n\}$, $\#B_0 = n$.

STEP 1: $v_1 \in V = span(w_1, w_2, \cdots w_n) \Leftrightarrow \{w_1, w_2 \dots w_n, v_1\}$ L.D. $\Leftrightarrow \{v_1, w_1, w_2 \cdots w_n\}$ L.D.

$\Leftrightarrow \exists j_1, j_1 \geq 1$. SMALLEST $w_{j_1} \in span(v_1, w_1, \cdots w_{j_1 - 1})$. st $v_1 = \lambda \cdot w_{j_1}$ $\lambda \in F$.

$V = span(w_1, w_2, \cdots w_n) = span(v_1, w_1, \cdots w_n) = span(v_1, w_1 \cdots w_{j_1-1}, w_{j_1+1}, \cdots w_n)$

$B_1 = \{v_1, w_1 \cdots w_{j_1-1}, w_{j_1+1}, \cdots w_n\}$, $\#B_1 = n$.

STEP 2: $v_2 \in V = span(v_1, w_1, w_2 \cdots w_{j_2} \cdots w_n) \Leftrightarrow \{v_1, w_1 \dots w_{j_2} \dots w_n, v_2\}$ LD. $\Leftrightarrow \{v_1, v_2, w_1 \dots w_{j_2} \dots w_n\}$ LD.

$\Leftrightarrow \exists j_2, j_2 \geq 2$. st $w_{j_2} \in span(v_1, v_2, w_1 \cdots w_{j_2 - 1})$. $\#B_2 = n$.

$\Rightarrow V = span(B_1) = span(v_1, v_2, w_1 \cdots w_{j_2} \dots w_n) = span(\underbrace{v_1, v_2 \cdots w_{j_2-1}, w_{j_2+1} \cdots w_n}_{B_2})$.

⋮

Step m: $v_m \in V = span(B_{m-1}) = span(v_1, v_2 \cdots \overbrace{v_m, w_1 \cdots w_{j_m-1}, w_{j_m+1}, \cdots w_n}^{n})$.

$\Leftrightarrow \{\underset{\substack{\Downarrow \\ LI}}{v_1, v_2} \cdots v_m, w_1, w_2 \cdots w_{j_1}, \underset{\substack{\Downarrow \\ \text{cannot be empty}}}{w_{j_2}}, \cdots \underline{\underline{w_{j_m}}}, \dots w_n\}$ LD. $\Rightarrow m < \#(\{v_m\} \cup B_{m-1}) = 1 + n$

$\Leftrightarrow m \le n$.

4.2. CLASS 10.

last class: we get CARDINALITY OF ANY SPANNING SET.

$\{v_1, v_2 \cdots v_m\}$ LI, $V = span(w_1, w_2, \cdots w_n) \Longrightarrow m \le n$.

ADDICATION:

All subspaces of R^2 are $\{(0,0)\}$, R^2 and $L = \{\lambda(a,b), (a,b) \in R^2, (a,b) \neq 0, \lambda \in R\}$.

First we check L are subspace: ....

Let S be a subspace.

S subspace $\Rightarrow (0,0) \in S \Rightarrow \{(0,0)\} \subseteq S$. If $\{(0,0)\} = S \Rightarrow$ We have done.

If $\{(0,0)\} \subsetneq S \Rightarrow \exists (a,b) \in R^2, (a,b) \neq (0,0) : (a,b) \in S$.

S subspace, $(a,b) \in S \Rightarrow \lambda(a,b) \in S, \forall \lambda \in R \Rightarrow l = \{\lambda(a,b), \lambda \in R\} \subseteq S$.

If $S = \{\lambda(a,b), \lambda \in R\}$ We have done.

If $\{\lambda(a,b), (a,b) \neq (0,0), \lambda \in R\} \subsetneq S \Rightarrow \exists (c,d) \in S$. $(c,d) \neq \lambda(a,b)$ $\forall \lambda \in R$.

$\Rightarrow \{(a,b), (c,d)\}$ is L.I

$R^2 = \{(x,y) = x(1,0) + y(0,1), x,y \in R\} = span((1,0),(0,1))$. span $((1,0),(0,1))$. LI.

$\Rightarrow (x,y) \in span((a,b),(c,d)) = \{\lambda_1(a,b) + \lambda_2(c,d), \lambda_1, \lambda_2 \in R\} \subseteq S$

$\Rightarrow R^2 \subseteq S \Rightarrow R^2 \subseteq S \subseteq R^2 \Rightarrow S = R^2$.

THEO: LET V BE A FINITE-DIMENTIONAL VECTOR SPACE, AND LET S BE A SUBSPACE OF V. THEN S IS ALSO A FINITE DIMENTIONAL VECTOR SPACE.

Proof: V finite-dimentional $\Leftrightarrow \exists$ finit set $X \subseteq V : V = \mathrm{span}(X)$, $X = \{w_1, w_2, \cdots w_n\}$

If $S = \{0\} = \mathrm{span}\{\emptyset\} \Rightarrow S$ is finite-dimentional.

if not $\exists v_1 \in S$ $v_1 \neq 0$.

if $S = \mathrm{span}(v_1) \Rightarrow S$ is finite-dimentional

If not, $\mathrm{span}(v_1) \subsetneq S \Rightarrow \exists v_2 \in S$, $v_2 \notin \mathrm{span}(v_1) \Longrightarrow \{v_1, v_2\}$ L.I.

If $S = \mathrm{span}(v_1, v_2) \Rightarrow S$ is FD.

If not $\mathrm{span}(v_1, v_2) \subsetneq S$, $\exists v_3 \in S$, $v_3 \notin \mathrm{span}(v_1, v_2) \Rightarrow \{v_1, v_2, v_3\}$ L.I.

Since the CARDINALITY of any L.I. set is $\leq n$. This algorial should stop in step k:

$S = \mathrm{span}(v_1, v_2 \ldots v_k) \Rightarrow S$ is F-D. for $k \leq n$.

BASIS.

Def:

A basis for a vector space V is an ordered L.I. spanning set. That is:

B is a basis $\Leftrightarrow$ B is L.I. and $V = \mathrm{span}(B)$.

REMARK: $B = \{v_1, v_2, \ldots v_n\}$ $V = \mathrm{span}(v_1, v_2, \ldots v_n)$. $\Leftrightarrow$ ANY $v \in V$ CAN BE WRITTEN AS A L.C. OF $v_1, v_2 \ldots v_n$: $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = \mu_1 v_1 + \mu_2 v_2 + \cdots + \mu_n v_n \Leftrightarrow 0 = (\lambda_1 - \mu_1)v_1 + (\lambda_2 - \mu_2)v_2 + \cdots + (\lambda_n - \mu_n)v_n$.

B is L.I. $\Leftrightarrow$ v can be written in a UNIQUE WAY.

Example:

1) $\mathbb{R}^3 = \{(x, y, z) \underset{\Downarrow}{=} x(1,0,0) + y(0,1,0) + z(0,0,1), x, y, z \in \mathbb{R}\}$.

can be written in a UNIQUE WAY.

$\Rightarrow E_3 = e_3 = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis.

$B = \{(0,1,0), (0,0,1), (1,0,0)\}$ is another basis.

$(x, y, z) = y(0,1,0) + z(0,0,1) + x(1,0,0)$

2) $S = \{(x, x+y, x), x, y \in \mathbb{R}\}$ is a subspace.

Find a basis:

spanning list.

$(x, x+y, x) = (x, x, x) + (0, y, 0) = x(1,1,1) + y(0,1,0)$. $S = \mathrm{span}((1,1,1), (0,1,0))$.

$a(1,1,1) + b(0,1,0) = (0,0,0) \Rightarrow a = b = 0$. $\Rightarrow$ L.I. SET.

$B = \{(1,1,1), (0,1,0)\}$ is a basis for S.

3) $\mathbb{R}_3[X] = \{p(x) \in \mathbb{R}[x] : p(x) = 0 \text{ or } \deg(p(x)) \leq 3\} = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$.

$= \mathrm{span}(1, x, x^2, x^3) \longrightarrow$ spanning list.

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0 \Leftrightarrow a_0 = a_1 = a_2 = a_3 = 0$ L.I.

$B = \{1, x, x^2, x^3\}$ is a basis of $\mathbb{R}_3[x]$.

4) $S = \mathrm{span}\big((1,0,0), (2,0,0), (1,1,0), (2,2,0)\big) = \mathrm{span}\big((1,0,0), (1,1,0), (2,2,0)\big) = \mathrm{span}\big((1,0,0) (1,1,0)\big)$.

So $B = \{(1,0,0), (1,1,0)\}$ is a basis of S.

Problem: HOW CAN WE CONSTRUCT BASIS FOR VECTOR SPACES?

THEO 1: any spanning list of V contains a basis.

$V = \mathrm{span}(X) \Rightarrow \exists B$ basis, $B \subseteq X$.

THEO 2: any L.I. set can be extended to a basis.

$Y \subseteq V$. Y is LI. $\Rightarrow \exists B$ basis, $Y \subseteq B$.

Proof: (We will write the proof only for finite-dimentional vector spaces).
(Not Finite-DIMENTIONAL, we use ZORN's LEMMA).

(1). Let V is F-D. X is finite, $X = \{w_1\, w_2 \dots w_n\} = B_0$.

STEP 1. If $w_1 = 0$, $B_1 = B_0 \setminus \{w_1\}$. If $w_1 \neq 0$ $B_1 = B_0$. $\Rightarrow \mathrm{span}(B_1) = \mathrm{span}(B_0)$.

STEP 2. if $w_2 \in \mathrm{span}(B_1) \Rightarrow B_2 = B_1 \setminus \{w_2\}$ if $w_2 \notin \mathrm{span}(B_1) \Rightarrow B_2 = B_1 \Rightarrow \mathrm{span}(B_2) = \mathrm{span}(B$

…

STEP n: If $w_n \in \mathrm{span}(B_{n-1}) \Rightarrow B_n = B_{n-1} \setminus \{w_n\}$. $\Rightarrow B_n$ is a basis of V.
if $w_n \notin \mathrm{span}(B_{n-1}) \Rightarrow B_n = B_{n-1}$.

$V = \mathrm{span}(X) = \mathrm{span}(B_0) = \mathrm{span}(B_1) = \cdots = \mathrm{span}(B_n)$.

We can use INDUCTION to prove it.

$\mathrm{span}(B_n) = \mathrm{span}(B_0) = \mathrm{span}(X)$. $\forall n$

if $n=1$. $\mathrm{span}(B_0) = \mathrm{span}(B_1)$. IH. $\Rightarrow B_n$ is a spanning list.

Take $n-1$. $\Rightarrow \mathrm{span}(B_0) = \mathrm{span}(B_{n-1}) \overset{\checkmark}{=} \mathrm{span}(B_n)$.

Now, need to prove B_n is L.I.

Assume B_n is LD. $B_n = \{v_1\, v_2 \dots v_s\}$ LD $\Rightarrow \exists v_j$ st $v_j \in \mathrm{span}(v_1, v_2 \cdots v_{j-1}\, v_{j+1} \dots v_n)$.

But $v_j = w_k$. $w_k \in \mathrm{span}(v_1, \dots v_{j-1}) \subseteq \mathrm{span}(B_{k-1})$ CONTRADICTION to step k.

$\Rightarrow B_n$ LI.

COROLLARY: any vector space admit a basis.

Proof: $V = \mathrm{span}(V) \Rightarrow \exists B \subseteq V$, B is basis.

Now we prove theo 2.

V is F-D. $\Rightarrow \exists X$ finite: $V = \mathrm{span}(X)$ If Y is L.I. $\#Y \leq \#X$.

$\Rightarrow Y$ is finite.

If $V = \mathrm{span}(Y) \Rightarrow Y$ is spanning set and LI. $\Rightarrow B = Y$ Basis.

If not, $\mathrm{span}(Y) \subsetneq V$, $\exists v_1 \in V$ $v_1 \notin \mathrm{span}(Y)$. $\Rightarrow B_1 = Y \cup \{v_1\}$ Y is LI.

If $V = \mathrm{span}(B_1) \Rightarrow Y \subseteq B_1$ B_1 is basis.

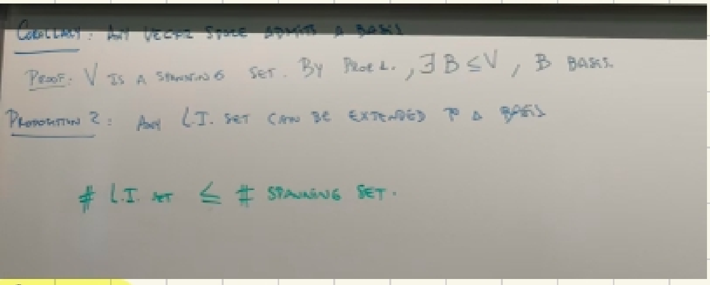
If not, $\mathrm{span}(B_1) \subsetneq V \Rightarrow \exists v_2 \in V$. $v_2 \notin \mathrm{span}(B_1) \Rightarrow B_2 = Y \cup \{v_1, v_2\}$ Y is L.I.

This has to finish in a finite number of steps since CARDINALITY of LI. sets $\leq \#X$.

Stop at Step k: $V = \mathrm{span}(B_k)$, B_k L.I. and spanning list $\Rightarrow Y \subseteq B_k$. B_k basis.

4.7. CLASS 11.

Proposition: Any spanning set contains a basis.



COROLLARY: ANY VECTOR SPACE ADMITS A BASIS

PROOF: V IS A SPANNING SET. BY PROP 1., $\exists B \subseteq V$, B BASIS.

PROPOSITION 2: ANY L.I. SET CAN BE EXTENDED TO A BASIS

$\#$ L.I. SET $\leq \#$ SPANNING SET.

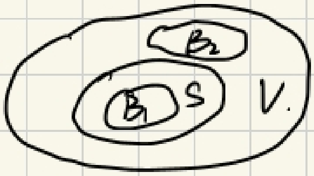
Corollary: Any subspace S of a vector space V admits a COMPLEMENT. That is, $\exists T \subseteq V$, a subspace st $V = S \oplus T$. (any $v \in V$ can be written as the addition $v = v_1 + v_2$, $v_1 \in S$, $v_2 \in T$.)

THEO:

S is a subspace of V $\Rightarrow$ S is a vector space $\Rightarrow \exists B_1$ a basis of S. $\Rightarrow B_1$ is LI in S.

$\Rightarrow B_1$ is LI in V. $\Rightarrow B_1 \subseteq S$. let B a basis of V.

$B = B_1 \cup B_2$. $B_2 = B \setminus B_1$



let $T = \text{span}(B_2)$ let's prove that $V = S \oplus T$.

$S + T = \text{span}(B_1) + \text{span}(B_2) = \text{span}(B_1 \cup B_2) = \text{span}(B) = V$.

$S \cap T = \{0\}$: Since $\{0\} \subseteq S \cap T$.

$v \in S \cap T \Rightarrow v \in S \wedge v \in T$, set $B_1 = \{v_i, i \in I\}$. $B_2 = \{w_j, j \in J\}$. $B = B_1 \cup B_2$. is a basis $\Rightarrow B_1 \cap B_2 = \phi$.

$v = a_1 v_1 + \cdots + a_k v_k = b_1 w_1 + \cdots + b_m w_m$. $\Rightarrow 0 = \underbrace{a_1 v_1 + \cdots + a_k v_k - b_1 w_1 - \cdots - b_m w_m}_{= B_1 \cup B_2.}$ Since $\underline{B_1 \cup B_2 \text{ is } LI}$.

$\Rightarrow a_1 = a_2 = \cdots = b_1 = b_2 = \cdots = b_m$. $\Rightarrow v = 0$.

DIMENTION

THEO: let B_1, B_2 be two basis of the F-vector space V. Then $\#B_1 = \#B_2$.

Proof: Basis = LI + spanning set.

B_1 LI, B_2 spanning set $\Rightarrow \#B_1 \leq \#B_2$.

B_2 LI, B_1 spanning set $\Rightarrow \#B_1 \geq \#B_2$ $\Rightarrow$ Then $\#B_1 = \#B_2$.

Def: $\dim_F V = \#B$. For B any basis for V, called the dimention of V.

Example.

1) F^n, $\{e_1, e_2, \ldots e_n\}$ is a basis $e_i = (0, 0, \cdots 1, 0, 0 \cdots 0)$. $\Rightarrow \dim_F F^n = n$.

2) $M_{n \times m}(F)$, $\{E_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis $E_{ij} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$. $\dim_F M_{n \times m}(F) = n \times m$.

3) $\dim_C C^2 = ?$ $\dim_R C^2 = ?$

$C^2 = \text{span}((1,0), (0,1))$, L.I.

$\Rightarrow \dim_C C^2 = 2$.

$C^2 = \text{span}((1,0), (i,0), (0,1), (0,i))$ $(a+bi, c+di) = a(1,0) + b(i,0) + c(0,1) + d(0,i)$.

$\{(1,0), (0,1), (i,0), (0,i)\}$ is LI in R.

$\Rightarrow \dim_R C^2 = 4$.

THEO 1: If S is a subspace of V. Then $\dim_F S \leq \dim_F V$.

Proof: let B_1 be a basis of S. B_2 a basis of V.

B_1 is LI in $S \Rightarrow B_1$ is LI in V.

B_2 is a spanning set for V. $\Rightarrow \# B_1 \leq \# B_2$. $\Rightarrow \dim_F S \leq \dim_F V$.

THEO: Let V be an F-vector space of finite dimention, that is $\dim_F V = n$. let $B \subseteq V$.
$\dim_F V = n$. then following are equivalent: $\# B = n$

a) B is a basis of V.

b) B is LI in V.

c) B is a spanning set for V.

Remark: false for infinite dimention:

$\{x, x^2, \dots\}$ is LI in $R[x]$ but is not a spanning set.

Proof:

a) $\Rightarrow$ b) ✓. Basis $\Leftrightarrow$ LI + spanning $\Longrightarrow$ LI.

b) $\Rightarrow$ c): B is LI in V, $\# B = n$. $\exists B_1$ basis: $B \subseteq B_1$ $\# B = n = \dim_F V = \# B_1$
$\Rightarrow B = B_1 \Rightarrow B$ spanning set.

c) $\Rightarrow$ a): B is a spanning set $\Rightarrow \exists B_2$ basis : $B_2 \subseteq B$.
$\# B_2 = \dim_F V = n = \# B$. $\Rightarrow B_2 = B \Rightarrow B$ is a basis.

Example:

i) $\dim_{\mathbb{C}} \mathbb{C}^2 = 2$. is $\{(1+i, 2), (2i, 3)\}$ a basis?

$(a+bi)(1+i, 2) + (c+di)(2i, 3) = ((a-b) + (a+b)i,\ 2a+2bi) + (-2d+2ci,\ 3c+3di)$
$= ((a-b-2d) + (a+b+2c)i,\ 2a+3c+(2b+3d)i) = (0,0) \Leftrightarrow \begin{cases} a-b-2d=0 \\ a+b+2c=0 \\ 2a+2c=0 \\ 2b+3d=0 \end{cases} \Rightarrow a=b=c=d=0 \Rightarrow$ is basis.

2) $B = \{(1, a_2, a_3), (0, 1, b_3), (0, 0, 1)\}$ is a basis F^3 since $\dim_F F^3 = 3$.

$\lambda_1(1, a_2, a_3) + \lambda_2(0, 1, b_3) + \lambda_3(0,0,1) = (0,0,0)$

$\Rightarrow (\lambda_1, \lambda_1 a + \lambda_2, \lambda_1 a_3 + \lambda_2 b_3 + \lambda_3) = (0,0,0) \Leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$.

B is LI, $\# B = 3 \Rightarrow B$ is a basis.

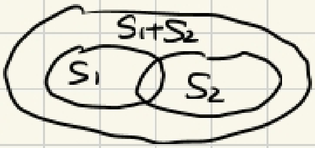
3) $B = \{1+ax+bx^2, x+cx^2, x^2\}$ LI in $R_{\leq 2}[x]$ $\dim_R R_{\leq 2}[x] = 3$.

$\Rightarrow B$ is a basis.

PROBLEM: Compute $\dim V = ?$

THEO: If $V = S_1 + S_2$, $\dim V < \infty$ Then $\dim(S_1+S_2) = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2)$

In particular: $\dim(S_1 \oplus S_2) = \dim S_1 + \dim S_2$.



Proof.

let $\{v_1, v_2 \cdots v_r\}$ be a basis for $S_1 \cap S_2 \Rightarrow \{v_1, v_2 \cdots v_r\}$ is LI in S_1 and S_2.

We can extend this set to basis for S_1 and S_2.

$B_1 = \{v_1 \dots v_r, u_1, \dots u_s\}$ basis for S_1. $\rightarrow \dim S_1 = r+s$. $\Rightarrow$ We have to prove that:

$B_2 = \{v_1 \cdots v_r, w_1 \cdots w_k\}$ basis for S_2. $\rightarrow \dim S_2 = r+k$. $\dim S_1+S_2 = r+s+r+k-r = r+s+k$.

let's prove that $B = \{v_1 \cdots v_r, w_1 \cdots w_k, u_1 \cdots u_s\}$ is a basis of S_1+S_2.

$span(B) = span(B_1 \cup B_2). = span(B_1) + span(B_2). = S_1 + S_2. \Rightarrow B$ is a spanning set.

B is LI:

$\in S_1 \qquad\qquad\qquad \in S_2.$

$0 = a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s + c_1w_1 + \cdots + c_kw_k. \Leftrightarrow a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s = (-c_1)w_1 + \cdots + (-c_k)w_k$ $= v \in S_1 \cap S_2. \Rightarrow (-c_1)w_1 + (-c_2)w_2 + \cdots (-c_k)w_k \in S_1 \cap S_2. = \lambda_1v_1 + \lambda_2v_2 + \cdots \lambda_rv_r$ Since $\{v_1, \cdots v_r\}$ is a basis of $S_1 \cap S_2$. So $0 = \lambda_1v_1 + \lambda_2v_2 + \cdots + \lambda_rv_r + c_1w_1 + c_2w_2 + \cdots + c_kw_k$. B_2 is LI. $\Rightarrow c_1 = c_2 = \cdots = c_k = \lambda_1 = \cdots = \lambda_r = 0. \Rightarrow a_1v_1 + a_2v_2 + \cdots + a_rv_r + b_1u_1 + b_2u_2 + \cdots + b_su_s = 0.$ B_1 is LI. $\Rightarrow a_1 = \cdots = a_r = b_1 = \cdots = b_s = 0.$

So B is LI.

4.9. CLASS 12.

Question: $S_1 + S_2 = ?$

$S_1 + S_2 \subseteq R^4$. $\dim S_1 + S_4 = 4 \Rightarrow R_1 + R_2 = R^4$.

THEO: if $S \subseteq V$ subspace, $\dim V < \infty \wedge \dim S = \dim V \Rightarrow S = V$.

Proof: let B be a basis of S. $\Rightarrow B$ is LI in S. $\Rightarrow B$ LI in V. $\#B = \dim S = \dim V. \Rightarrow B$ is a basis in V. $\Rightarrow S = span(B) = V$.

Remark: THE PREVIOUS THEO IS NOT TRUE WITHOUT THE ASSUMPTIONS:

1) $S \nsubseteq V$. $S = \{(x,0), x \in R\}$, $V = \{(0,y), y \in R\}$, $\dim S = \dim V = 1$. But $S \neq V$.

2) $\dim V = \infty$: $S = \{p(x) \in R(x) : p(0) = a_0 = 0\} \subsetneq R[x] = V = \{1, x, x^2, \dots\}$. $\dim S = \dim V = \infty$. But $S \neq V$

3) $\dim S \neq \dim V$: $S = \{(x,0) : x \in R\} \subsetneq R^2$. $\dim R^2 = 2 < \infty$. But $S \neq R^2$.

Matrices.

$M_{n\times m}(F)$ = set of matrices of order $n \times m$ with coefficients in F.

$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ A_{n1} & \cdots & a_{nm} \end{pmatrix}$ = matrix in $M_{n\times m}(F)$ = set of $n \times m$ elements in F. Arranged in m ROWS and n COLUMNS.

A_{ij} = coefficient in ROW i, COLUMN j.

$R_i = R_i(A) = i$-ROW $= [A_{i1}\ A_{i2}\ A_{i3} \cdots A_{im}]$. $C_j = C_j(A) = \begin{bmatrix} A_{1j} \\ A_{2j} \\ A_{nj} \end{bmatrix}$

$A \in M_{n\times m}(F)$ then $A = B \Leftrightarrow (n,m) = (s,t)$. and $A_{ij} = B_{ij}$ $\forall i \in [1,n]$ $\forall j \in [1,m]$.

Operations:

ADDITIONS: $M_{n\times m}(F) \times M_{n\times m}(F) \xrightarrow{+} M_{n\times m}(F).$

$(A,B) \longrightarrow A + B = (A+B)_{ij} = A_{ij} + B_{ij}.$

PROPERIES: (S_1) ASSOCIATIVE. (S_2) COMMUTATIVE.

(S_3) $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

(S_4) $-\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} -a_{11} & \cdots & -a_{1m} \\ \vdots & & \\ -a_{n1} & \cdots & -a_{nm} \end{pmatrix}.$

Product by scalars: $F \times M_{n\times m}(F) \longrightarrow M_{n\times m}(F)$.

$$(\lambda, A) \longrightarrow (\lambda A). = \lambda\cdot A_{ij}.$$

Properies:

1) $\lambda\cdot(A+B) = \lambda A+\lambda B \in M_{n\times m}(F)$.

$(\lambda(A+B))_{ij} = \lambda\cdot(A+B)_{ij} = \lambda(A_{ij}+B_{ij}) = \lambda\cdot A_{ij}+\lambda\cdot B_{ij} = (\lambda A+\lambda B)_{ij}$. $\forall ij$.

2) $(\lambda+\mu)A = \lambda A+\mu A$.

3) $(\lambda\cdot\mu)A = \lambda(\mu A)$.

4) $1_F\cdot A = A$. $\forall A$.

⇒ $M_{n\times m}(F)$ is F- vector space:

$M_{n\times m}(F)\times M_{m\times s}(F) \longrightarrow M_{n\times s}(F)$. (if $n=m=s$: $M_n(F) = M_{n\times n}(F)$.).

$M_n(F)\times M_n(F) \xrightarrow{\cdot} M_n(F)$

$(A,B) \xrightarrow{\cdot} A.B$. $\quad (A\cdot B)_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj} = \langle R_i(A), C_j(B)\rangle$.

Properties:

P1. **Associativity:** $(\overset{n\times m}{A}\cdot\overset{m\times s}{B})\overset{s\times t}{C} = \overset{n\times m}{A}(\overset{m\times s}{B}\overset{s\times t}{C})$ $\quad ((A\cdot B)C)_{ij} = \cdots = (A(BC))_{ij}$.

P3. **Identity:** $\overset{n\times n}{I_n}\cdot\overset{n\times m}{A} = \overset{n\times m}{A} = \overset{n\times m}{A}\cdot\overset{m\times m}{I_m}$.

$I_s = \begin{bmatrix}1&\cdots&0\\0&1&\vdots\\\vdots&\ddots&\vdots\\0&0\cdots&1\end{bmatrix}$ (In $M_n(F)$, $I = I_n$).

Distributivity: $\overset{n\times m}{A}\cdot(\overset{m\times s}{B}+\overset{m\times s}{C}) = \overset{n\times m}{A}\cdot\overset{m\times s}{B} + \overset{n\times m}{A}\cdot\overset{m\times s}{C}$.

1) $\lambda\cdot(A\cdot B) = (\lambda\cdot A)\cdot B = A\cdot(\lambda B)$.

2) $\lambda\cdot A = (\lambda\cdot Id)A = A\cdot(\lambda\cdot Id)$.

Remark: (P2) Commutivity is not true.

$a\neq b$: $\begin{pmatrix}a&0\\0&b\end{pmatrix}\cdot\begin{pmatrix}x&y\\z&t\end{pmatrix} = \begin{pmatrix}ax&ay\\bz&bt\end{pmatrix}$. $\qquad \begin{pmatrix}x&y\\z&t\end{pmatrix}\cdot\begin{pmatrix}a&0\\0&b\end{pmatrix} = \begin{pmatrix}ax&by\\az&bt\end{pmatrix}$ not equal.

P4 A. $\exists A^{-1}$: $A\cdot A^{-1} = Id$. NOT TRUE.

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} = I_2 = \underset{A}{\begin{pmatrix}1&1\\0&0\end{pmatrix}}\underset{A^{-1}}{\begin{pmatrix}x_1&x_2\\y_1&y_2\end{pmatrix}} = \underset{\neq I_2.}{\begin{pmatrix}x_1&x_2\\0&0\end{pmatrix}} \Rightarrow \nexists A^{-1}.$$

$A\cdot B = 0 \not\Rightarrow A=0$ or $B=0$.

$$\underset{\substack{\neq\\0}}{\begin{pmatrix}0&1\\0&0\end{pmatrix}}\cdot\underset{\substack{\neq\\0.}}{\begin{pmatrix}1&2\\0&0\end{pmatrix}} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

ELEMENTARY OPERATIONS ON ROWS. = ELEMENTARY MATRICES.

element operations:

1) $R_i \longleftrightarrow R_j$ interchange Rows.

2) $R_i \longrightarrow a \cdot R_i$ $a \neq 0$. multiplication with a.

3) $R_i \longrightarrow R_i + aR_j$ $i \neq j$. Replace R_i by $R_i + aR_j$.

1) $P_{ij} = \begin{bmatrix} 1 & \cdots & & & 0 \\ 0 & \ddots & & & \\ \vdots & 0 & \cdots & 1 & \\ & 1 & 1 & 0 & \\ 0 & \vdots & & \ddots & 1 \\ & i & & j & \end{bmatrix}$

$R_i \leftrightarrow R_j$: $P_{ij} \times A$. $Id \longrightarrow P_{ij}$.

$C_i \leftrightarrow C_j$: $A \times P_{ij}$. $Id \rightarrow P_{ij}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

2) $M_i(a) = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & a & \vdots \\ \vdots & & \ddots \\ 0 & \cdots & 1 \end{bmatrix}$

$R_i \longrightarrow a \cdot R_i$ $M_i(a) \cdot A$. $Id \longrightarrow M_i(a)$

$C_i \longrightarrow a \cdot C_i$ $A \cdot M_i(a)$. $Id \longrightarrow M_i(a)$. ※

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3) $T_{ij}(a) = \begin{array}{c} \\ i \\ j \\ \\ \end{array} \begin{bmatrix} 1 & \cdots & & & 0 \\ \vdots & 1 & \cdots & a & \vdots \\ & & \ddots & 1 & \\ 0 & & & \ddots & 1 \\ & i & & j & \end{bmatrix}$

$R_i \longrightarrow R_i + aR_j$. $T_{ij}(a) \cdot A$. $Id \longrightarrow T_{ij}(a)$

$C_j \longrightarrow C_j + aC_i$. $A \cdot T_{ij}(a)$ $Id \longrightarrow T_{ij}(a)$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & a \\ 0 & 0 & 1 \end{pmatrix}.$$

$R_i = (a_{i1}\ a_{i2} \cdots 0$.

$R_j = (a_{j1}\ a_{j2}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Properties:

1) $P_{ij} \cdot P_{ij} = P_{ij}(P_{ij} \cdot Id) = Id$.

$\Rightarrow P_{ij} \cdot P_{ij} \cdot A = A$.

2) $M_i(a) \cdot M_i(a^{-1}) = Id$.

3) $T_{ij}(-a) \cdot T_{ij}(a) = Id$.

Example to prove the properties:

$Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{R_2 \leftrightarrow R_3}{\longleftrightarrow} P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$: $P_{23} \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$.

$Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 \rightarrow R_2 + 5R_1]{} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: $T_{21}(5) \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{11}+a_{21} & 5a_{12}+a_{22} & 5a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$P_{ij} \cdot P_{ij} \cdot A = P_{ij} \cdot P_{ij} \cdot \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{bmatrix} = P_{ij} \cdot \begin{bmatrix} R_1 \\ \vdots \\ R_j \\ R_i \\ \vdots \end{bmatrix} = \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ R_j \\ \vdots \end{bmatrix}$.

$T_{23}(-4) \cdot T_{23}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$R_2 \longrightarrow R_2 + (-4)R_3$ $R_2 + (-4)R_3 + 4R_3 \longrightarrow R_2$.

$A \cdot T_{23}(4) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a_{11} & a_{12} & a_{13}+4a_{12} \\ a_{21} & a_{22} & a_{23}+4a_{22} \\ a_{31} & a_{32} & a_{33}+4a_{32} \end{pmatrix}$

Def: $A, B \in M_{n\times m}(F)$. We say that A and B are ROW-EQUIVALENT if there exists a finite number of ELEMENTARY MATRICES, $E_1 E_2 E_3 \dots E_s$. st

$$E_s \cdots E_1 \cdot A = B.$$

Remark: THIS RELATION IS AN EQUIVALENCE RELATION.

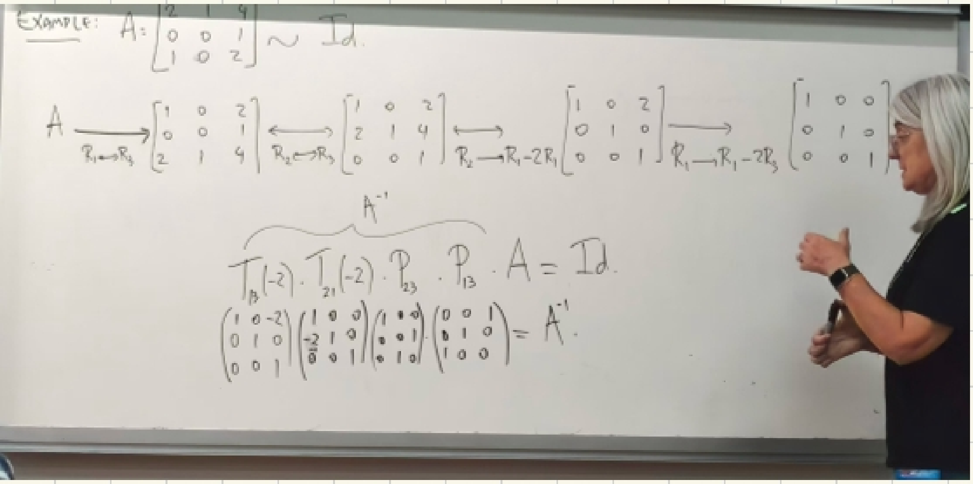
Reflexive: $Id = M_i(1)$: $A \sim A$ since $M_i(1)\cdot A = Id \cdot A = A$.

SYMMETRIC: $P_{ij}^{-1} = P_{ij}$ $M_i(a)^{-1} = M_i(a^{-1})$, $T_{ij}(a)^{-1} = T_{ij}(-a)$.

$$A \sim B \Leftrightarrow E_s \cdots E_1 A = B \Leftrightarrow A = E_s^{-1} \cdots E_1^{-1} B \Leftrightarrow B \sim A.$$

TRANSITIVITY. $A \sim B$, $B \sim C \Leftrightarrow E_s \cdots E_1 \cdot A = B$ $E_k \cdots E_1' \cdot B = C$. $\Rightarrow E_k \cdots E_1' E_s \cdots E_1 A = C$.

$\Leftrightarrow A \sim C$.



EXAMPLE: $A: \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim Id.$

$$A \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix} \xleftrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \xleftrightarrow[R_2 \longrightarrow R_1 - 2R_1]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \longrightarrow R_1 - 2R_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overbrace{T_{13}(-2) \cdot T_{21}(-2) \cdot P_{23} \cdot P_{13}}^{A^{-1}} \cdot A = Id.$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A^{-1}.$$

CLASS 13.

ROW \ COLUMN.

1) Interchange rows R_i and R_j.
2) Multiply R_i by a non-zero scalar of $\alpha \in F$.
3). Replace R_i by $R_i + aR_j$. $i \neq j$.

(Row) ELEMENTARY MATRICES.

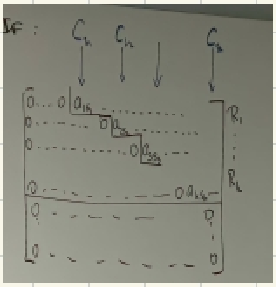
1) $Id \xrightarrow{R_i \leftrightarrow R_j} P_{ij} \xrightarrow{R_j \leftrightarrow R_i} Id$. $A \to P_{ij} \cdot A$.
2) $Id \xrightarrow{R_i \to R_i \cdot a} M_i(a) \xrightarrow{\cdot a^{-1}} Id$. $A \to M_i(a) \cdot A$.
3) $Id \xrightarrow{R_i \to R_i + aR_j} T_{ij}(a) \xrightarrow{-a \cdot R_i} Id$. $A \to T_{ij}(a) \cdot A$.

COL. $A \to A \cdot P_{ij}$. $A \to A \cdot M_i(a)$. $A \to A \cdot T_{ij}(a)$.

Def: $A . B \in M_{n\times m}(F)$ are row equivalent if $\exists E_1 . E_2 \dots E_s$ elementary matrices st $E_1 E_2 \cdots E_s \cdot A = B$.

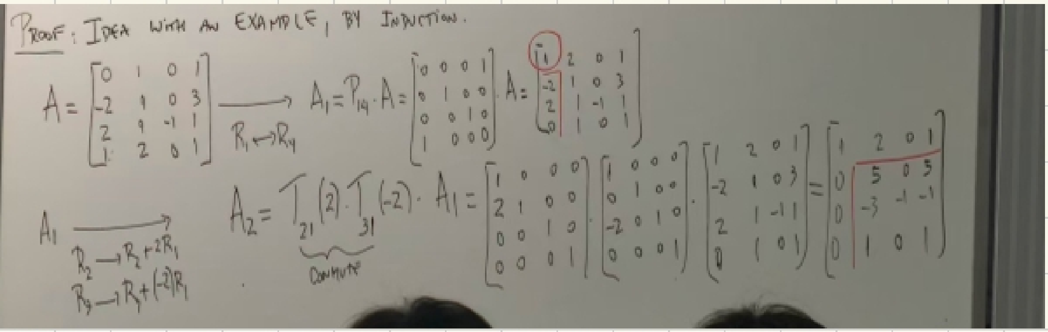
Def: A matrices $A \in M_{n\times m}(F)$ is called Row Echelon if:

1) The non-zero appears first. $R_1 . R_2 \cdots R_k$ non-zero. $R_{k+1} \cdots R_n$ is 0.
2) If the first non-zero element in R_i appears in Column S_i. Then $S_1 < S_2 \cdots < S_k$
3) $a_{is_i} = 1$. $1 \le i \le k$.



Def: A matrix A is called Row-Reduced Echelon if it is Row Echelon and each column C_{s_i} has all its elements equal 0 except $a_{is_i} = 1$.

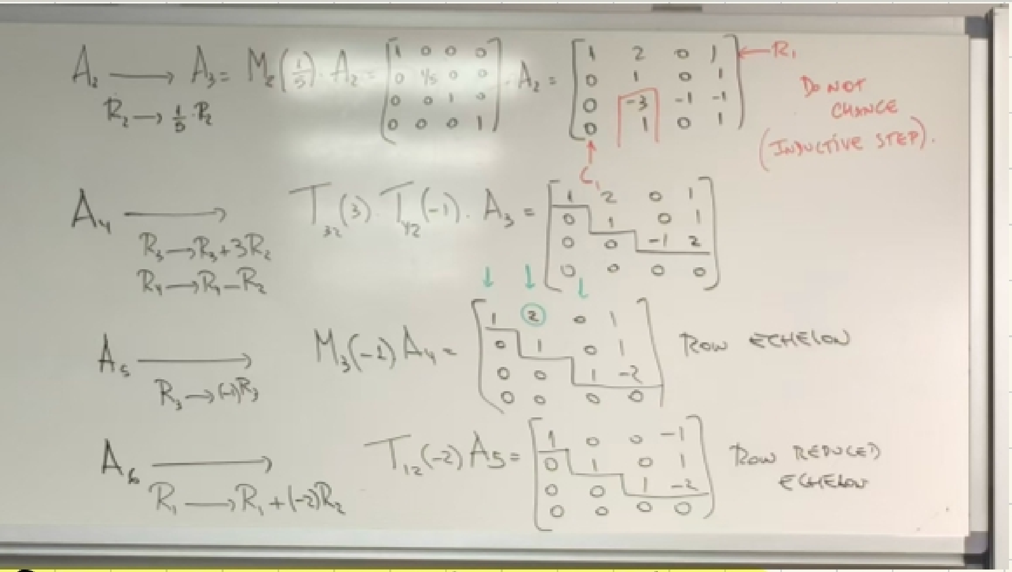
THEO: If $A \in M_{m \times m}(F)$, then exists $E_1 E_2 \cdots E_k$ elementary matrices st. $E_1 \cdots E_k A$ is Row-Reduced Echelon.



PROOF: IDEA WITH AN EXAMPLE, BY INDUCTION.

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -2 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_4]{} A_1 = P_{14} \cdot A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot A = \begin{bmatrix} \bigcirc\!\!\!\!1 & 2 & 0 & 1 \\ -2 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$A_1 \xrightarrow[\substack{R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + (-2)R_1}]{} A_2 = \underbrace{T_{21}(2) \cdot T_{31}(-2)}_{\text{COMMUTE}} \cdot A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 & 1 \\ -2 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 5 & 0 & 5 \\ 0 & -3 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$



$$A_2 \xrightarrow[R_2 \to \frac{1}{5} R_2]{} A_3 = M_2\left(\tfrac{1}{5}\right) \cdot A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot A_2 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & -1 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \leftarrow R_1$$

C_1

DO NOT CHANGE (INDUCTIVE STEP).

$$A_4 \xrightarrow[\substack{R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - R_2}]{} T_{32}(3) \cdot T_{42}(-1) \cdot A_3 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_5 \xrightarrow[R_3 \to (-1)R_3]{} M_3(-1) A_4 = \begin{bmatrix} 1 & \textcircled{2} & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ROW ECHELON

$$A_6 \xrightarrow[R_1 \to R_1 + (-2)R_2]{} T_{12}(-2) A_5 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ ROW REDUCED ECHELON

SYSTEMS OF LINEAR EQUATIONS

A system of n Linear equation in m unknows is

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1m} x_m = b_1 \\ \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nm} x_m = b_m \end{cases}$$

$a_{ij}, b_i \in F$. are the coefficients,
$x_1, x_2 \ldots x_m$ unknows.

Problem: SOLVE THE SYSTEM: Find all $(x_1, x_2, \ldots x_m) \in F^m$ that satisfy the system.

Remark:

WE CAN USE MATRICES AND PRODUCT OF MATRICES TO REPRESENT THE SYSTEM.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \Rightarrow A \cdot x = b.$$

S = solutions for (*) $= \{(x_1, \ldots x_m) \in F^n : Ax = b\} \subseteq F^n$.

Def: A SYSTEM $Ax = b$ IS CALLED:

1) HOMOGENEOUS if $b = 0$.

2) **INCONSISTENT** if $S = \emptyset$.

3) **CONSISTENT** if $S \neq \phi$

4) **CONSISTENT INDEPENDENT** if $\#S = 1$.

DEPENDENT if $\#S > 1$.

Remark:

1) Any Homogeneous system is Consistent since $(x_1, x_2, \dots x_m) = (0, 0, \dots 0)$.

2) The set of solutions of any Homogeneous system is a subspace of F^n.

3) If $b \neq 0$, S is not a subspace, since $(0, 0, \dots 0) \notin S$.

Proof (2):

① $(0,0,0 \dots 0) \in S = \{(x_1\, x_2 \dots x_n) \in F^n : Ax = 0\} \subseteq F^n$

② $Ax_1 = 0$. $Ax_2 = 0$. $\Rightarrow A(x_1 + x_2) = Ax_1 + Ax_2 = 0$ $\Rightarrow x_1, x_2 \in S \Rightarrow x_1 + x_2 \in S$.

③ $Ax = 0$, $\lambda \in F$. $\Rightarrow A \cdot (\lambda x) = \lambda \cdot Ax = \lambda \cdot 0 = 0$. $x \in S$. $\lambda \in F \Rightarrow \lambda x \in S$.

THEO: Let $S = \{(x_1 \dots x_m) \in F^m : Ax = b\}$, $S_0 = \{(x_1 \dots x_m) \in F^m : Ax = 0\}$, and let $z_0 \in S$. then: $S = \{z_0 + w,\ w \in S_0\}$.

Proof:

$S \supseteq \{z_0 + w, w \in S_0\}$: $A(z_0 + w) = Az_0 + Aw = b + 0 = b$ ✓.

$S \subseteq \{z_0 + w. w \in S_0\}$: $z_1 \in S$. $w = z_1 - z_0$: $Aw = A(z_1 - z_0) = Az_1 - Az_0 = b - b = 0$. $\Rightarrow w = z_1 - z_0$.
$\Rightarrow z_1 = w + z_0$. ✓

Corollary: If S is a CONSISTENT DEPENDENT SYSTEM, then $\#F \leq \#S$.
In particular, if $\#F = \infty$, then the possibilities of S are: ① $\#S = 0$ ② $\#S = 1$ ③ $\#S = \infty$

Proof:

If S is CONSISTENT DEPENDENT, take $z_1, z_2 \in S$, $z_1 \neq z_2$. $\Rightarrow w = z_1 - z_2 \in S_0$, $w \neq 0$. Since S_0 is a subspace of F^n. then $\{\lambda w, \lambda \in F\} \subseteq S_0$, $\#\{\lambda w, \lambda \in F\} = \#F$. Since $F \longleftrightarrow \{\lambda w, \lambda \in F\}$. Bijection.
$\lambda \longleftrightarrow \lambda w$.
$\{z_0 + \lambda w, \lambda \in F\} \subseteq S$, $\#\{z_0 + \lambda w, \lambda \in F\} = \#F$. $\Rightarrow \#F \leq \#S$.

THEO. Let $Ax = b$ be a system of LINEAR EQUATIONS. Consider $[A \vdots b] \in M_{m \times (n+1)}(F)$ the matrix associated to the system. If $[A' \vdots b']$ is obtained from $[A \vdots b]$ by applying ROW operations. then:

$$S = \{(x_1, \dots x_m) \in F^m : Ax = b\} = S' = \{(x_1 \dots x_m) \in F^m : A'x = b'\}.$$

Proof:

$[A \vdots b] \xrightarrow{E_1} \xrightarrow{E_2} \dots \xrightarrow{E_k} [A' \vdots b'] \Leftrightarrow E_k \cdots E_1 [A \vdots b] = [A' \vdots b'] \Leftrightarrow \begin{cases} E_k \cdots E_1 A = A'. \\ E_k \cdots E_1 b = b'. \end{cases}$

$S \subseteq S'$: If $Ax = b \Rightarrow A'x = E_k \cdots E_1 Ax = E_k \cdots E_1 b \Rightarrow A'x = b'$. ✓.

$S \supseteq S'$: If $\begin{cases} E_k \cdots E_1 A = A' \\ E_k \cdots E_1 b = b' \end{cases} \Rightarrow \begin{cases} E_1^{-1} \cdots E_k^{-1} A' = A. \\ E_1^{-1} \cdots E_k^{-1} b' = b \end{cases} \Rightarrow Ax = E_1^{-1} \cdots E_k^{-1} . A'x = E_1^{-1} \cdots E_k^{-1} b' = b \Rightarrow Ax = b$.

$\Rightarrow S = S'$.

LINEAR MAPS. (F-vector spaces, V,W are F-vector spaces, want to compare them).

Def: A linear map from V to W is a function $T: V \to W$. st:

$T(v_1+v_2) = T(v_1)+T(v_2)$. $\forall v_1, v_2 \in V$.

$T(\lambda v) = \lambda \cdot T(v)$. $\forall \lambda \in F$. $v \in V$.

NOTATION: $\mathcal{L}_F(V,W) = \mathrm{Hom}_F(V,W)$ = SET OF All LINEAR MAPS FROM V to W. For V, W two vector spaces.

Example:

1) $T: V \to W$, $T(v) = 0_W$ is a linear map.

2). $Id: V \to V$ $Id(v) = v$. is ✓.

3) $T: \mathbb{C} \to \mathbb{C}$ $T(a+bi) = a-bi$ ⇒ $T \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$. but $T \notin \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$.

$\mathbb{C}$ is an $\mathbb{R}$-vector space $\lambda \in \mathbb{R}$: $T(\lambda(a+bi)) = T(\lambda a+\lambda bi) = \lambda a - \lambda bi$.

$\lambda T(a+bi) = \lambda a - \lambda bi$ $T \in \mathrm{Hom}_{\mathbb{R}}(V,W)$.

$\mathbb{C}$ is an $\mathbb{C}$-vector space. $\lambda \in \mathbb{C}$: $\lambda = x+iy$.

$T((x+iy)\cdot(a+bi)) = T((xa-yb)+(xb+ay)i) = xa-yb-(xb+ay)i$.

$(x+yi)\cdot T(a+bi) = (x+yi)(a-bi) = ax+by+(ay-bx)i$. not equal.

So $T \notin \mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C})$.

Example.

1) Describe the set $\mathrm{Hom}_F(F,F)$. $= \{T: F \to F,\ T(a) = aT(1).\ \text{for } T(1) \in F\}$.

$T: F \to F$.

$T(a) = T(a \cdot 1) = a \cdot T(1)$. T is uniquely determined by $T(1)$. ($\mathrm{Hom} \subseteq \{\ \}$).

let $T(1) = c. \in F$.

Define $T(a) = T(a\cdot 1) = a \cdot T(1) = a \cdot c$.

Check: T is a linear map. ⇒ ($\mathrm{Hom} \supseteq \{\ \}$).

$T(a+a') = (a+a')\cdot T(1) = (a+a')\cdot c = ac+a'c. = T(a)+T(a')$

$T(\lambda a) = \lambda a \cdot T(1) = \lambda a \cdot c = \lambda \cdot ac = \lambda \cdot T(a)$.

2) Describe $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) = \{T: \mathbb{C} \to \mathbb{C},\ T(a+bi) = a\alpha+b\beta,\ \text{for } \alpha, \beta \in \mathbb{C}\}$.

$T: \mathbb{C} \to \mathbb{C}$. $T(a+bi) = T(a)+T(bi). = T(a\cdot 1)+T(bi) = a\underbrace{T(1)}_{\alpha} + b\underbrace{T(i)}_{\beta}$

T is uniquely determined by $T(1)$, $T(i)$. ($\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) \subseteq \{\ \}$)

Now, check T is linear map.

$T(a+bi) = a\alpha + b\cdot\beta$

$T((a+bi)+(c+di)) = T((a+c)+(b+d)i) = (a+b)\alpha+(b+d)\beta$. $T(a+bi)+T(c+di) = a\alpha+b\beta+c\alpha+d\beta = \ldots$

$T(\lambda(a+bi)) = T(\lambda a+\lambda bi) = \lambda a\alpha+\lambda b\beta. = \lambda T(a+bi)$.

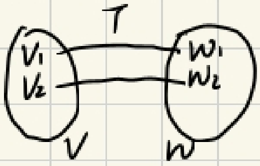
⇒ ($\mathrm{Hom} \supseteq \{\ \}$).

3) Describe the set $\mathrm{Hom}_F(F^2,W) = \{T: F^2 \to W : T(x,y) = x\alpha+y\beta.\ \text{for some } \alpha,\beta \in W\}$.

$T: F^2 \to W$. $T(x,y) = T(x(1,0)+y(0,1)). = xT(1,0)+yT(0,1)$. ⇒ T is uniquely determined by $T(1,0)$, $T(0,1)$. Now check it is LM. $\subseteq$

$T: F^2 \to W$ $T(x,y) = x\alpha+y\beta$. ... $\supseteq$

THEO: let $\{v_1 \dots v_n\}$ be basis of V. let $\{w_1 \dots w_n\}$ in W. $\exists! \, T: V \to W$ Linear Map st $T(v_i) = w_i$, $\forall i = 1 \dots n$.



Proof:

$\exists \, T(v) = T(a_1v_1 + a_2v_2 + \dots + a_nv_n) = a_1w_1 + a_2w_2 + \dots + a_nw_n$

(Well define, since $v = a_1v_1 + a_2v_2 + \dots + a_nv_n$. in a unique way). $\Rightarrow T$ is LM. =

$T(v+u) = T((a_1v_1 + \dots + a_nv_n) + (b_1v_1 + \dots + b_nv_n)) = T((a_1+b_1)v_1 + (a_2+b_2)v_2 + \dots + (a_n+b_n)v_n)$.

$= (a_1+b_1)w_1 + (a_2+b_2) + w_2 + \dots + (a_n+b_n)w_n = (a_1w_1 + a_2w_2 + \dots + a_nw_n) + (b_1w_1 + b_2w_2 + \dots + b_nw_n)$

$= T(v) + T(u)$.

$T(\lambda v) = T(\lambda(a_1v_1 + a_2v_2 + \dots + a_nv_n)) = \lambda a_1w_1 + \lambda a_2w_2 + \dots + \lambda a_nw_n = \lambda T(v)$.

Proof : $T(v_i) = w_i$.

$T(v_i) = T(0 \cdot v_1 + 0 \cdot v_2 + \dots + 1 \cdot v_i + \dots + 0 \cdot v_n) = 0 \cdot w_1 + 0 \cdot w_2 + \dots + 1 \cdot w_i + \dots + 0 \cdot w_n = w_i$.

let $T \in Hom_F(V, W)$ st $T(s_i) = w_i$ $\forall i = 1, \dots n$.

$T(v) = T(a_1v_1 + \dots + a_nv_n) = T(a_1v_1) + T(a_2v_2) + \dots + T(a_nv_n) = a_1w_1 + a_2w_2 + \dots + a_nw_n$.

Example.

There is no linear map $T: R^2 \to R^3$ st $T(1,0) = (1,2,3)$, $T(0,1) = (3,2,1)$. $T(1,1) = (0,1,0)$.

If $T \in Hom_R(R^2, R^3)$: $T(1,0) = (1,2,3)$. $T(0,1) = (3,2,1)$. $\Rightarrow T(x,y) = x \cdot T(1,0) + y T(0,1)$.

$= x \cdot (1,2,3) + y(3,2,1) = (x+3y, 2x+2y, 3x+y)$. $\Rightarrow T(1,1) = (4,4,4) \neq (0,1,0)$.

Example.

Find LM $T: R^4 \to R^2$. T is uniquely determined by $T(1,1,0,0) = w_1 = (a_1, b_1)$.

$T(0,1,0,0) = w_2 = (a_2, b_2)$, $T(0,0,1,0) = w_3 = (a_3, b_3)$. $T(0,0,0,1) = (a_4, b_4)$

$\Rightarrow T(x_1, x_2, x_3, x_4) = T(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4) = T(x_1e_1, 0,0,0) + T(0, x_2e_2, 0, 0) + T(0,0,x_3e_3, 0) + T(0,0,0,x_4e_4)$

$= x_1e_1w_1 + x_2e_2w_2 + x_3e_3w_3 + x_4e_4w_4 = (x_1e_1a_1 + x_2e_2a_2 + x_3e_3a_3 + x_4e_4a_4, x_1e_1b_1 + x_2e_2b_2 + x_3e_3b_3 + x_4e_4b_4)$.

Now, I want to define ALGEBRAIC STRUCTURES on the set $Hom_F(V, W)$.

ADDITION:

$$Hom_F(V,W) \times Hom_F(V,W) \xrightarrow{+} Hom_F(V,W).$$

$$(T_1, T_2) \longrightarrow T_1 + T_2 : V \to W.$$

$$(T_1 + T_2)(v) = T_1(v_1) + T_2(v_2)$$

- $T_1 + T_2 \in Hom_F(V,W)$: $(T_1+T_2)(v_1+v_2) \overset{Def}{=} T_1(v_1+v_2) + T_2(v_1+v_2) \overset{LM}{=} T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2)$
  $= (T_1+T_2)(v_1) + (T_1+T_2)(v_2)$
- $(T_1+T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v) = \lambda T_1(v) + \lambda T_2(v) = \lambda(T_1(v) + T_2(v)) = \lambda(T_1+T_2)(v)$.

Properties: Associativity: $(T_1+T_2)+T_3 = T_1+(T_2+T_3)$.
Communitivity: $T_1 + T_2 = T_2 + T_1$.
Indentity: $0 + T = T = T + 0$.
Inverse: T: $-T$ as $(-T)(v) = -T(v)$.

THEO: let $\langle v_1 \cdots v_n \rangle$ be a basis of V. let $\langle w_1 \cdots w_n \rangle$ of W. V, W F-vector space

Then $\exists!\ T: V \longrightarrow W$ F-linear map. st $T(v_i) = w_i \quad \forall i = 1, 2 \dots n$.

Proof: $\exists T(v) = T(a_1v_1+a_2v_2+\cdots+a_nv_n) = a_1w_1+a_2w_2+\cdots+a_nw_n$. $\Rightarrow$ Well define, since $v = a_1v_1+a_2v_2+\cdots+a_nv_n$

T is a LM: $T(v+u) = T(a_1v_1+a_2v_2+\cdots+a_nv_n+b_1v_1+\cdots+b_nv_n) = T((a_1+b_1)v_1+(a_2+b_2)v_2+\cdots+(a_n+b_n)v_n) = (a_1+b_1)w_1+\cdots+(a_n+b_n)w_n$

$T(\lambda v)$ is the same.

Product by scalars.

$F \times Hom_F(V,W) \longrightarrow Hom_F(V,W)$.

$(\lambda, T) \longrightarrow \lambda T: V \longrightarrow W$. $\quad (\lambda T)(v) = \lambda \cdot T(v)$.

λT is a LM.

$(\lambda \cdot T)(v_1+v_2) = \lambda \cdot T(v_1+v_2) = \lambda \cdot (T(v_1)+T(v_2)) = \lambda T(v_1) + \lambda T(v_2) = (\lambda T)(v_1) + (\lambda T)(v_2)$.

$(\lambda T)(\mu v) = \mu \cdot \lambda T(v) = \mu \cdot (\lambda T) \cdot (v)$

Properties.

$(\lambda_1+\lambda_2)T = \lambda_1 T + \lambda_2 T$.

$\lambda(T_1+T_2) = \lambda T_1 + \lambda T_2$.

$(\lambda_1 \cdot \lambda_2)T = \lambda_1 \cdot (\lambda_2 T)$

$1 \cdot T = T$. $\quad \Rightarrow Hom_F(V,W)$ is an F-vector space.

COMPOSITION OF LINEAR MAPS.

$Hom_F(V,W) \times Hom_F(W,U) \longrightarrow Hom_F(V,U)$.

$(f, g) \longrightarrow g \circ f: V \rightarrow W$.

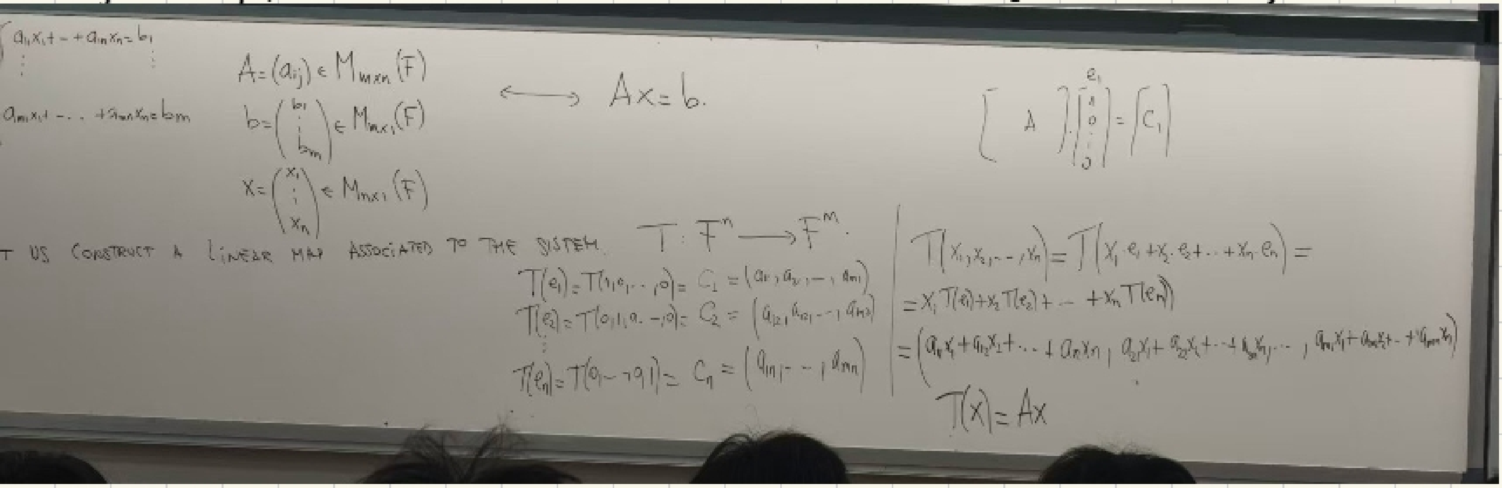
$g \circ f$ is a LM.

$(g \circ f)(v_1+v_2) = g(f(v_1+v_2)) = g(f(v_1)+f(v_2)) = g \circ f(v_1) + g \circ f(v_2)$.

$(g \circ f)(\lambda v) = g(f(\lambda v)) = g(\lambda f(v)) = \lambda \cdot g \circ f(v)$.

Properties.

1). $h \circ (g \circ f) = h \circ g \circ f$. $\quad$ 2). $f: V \rightarrow W$: $id_W \circ f = f = f \circ id_V$. $\quad$ 3). $g \circ (f_1+f_2) = g \circ f_1 + g \circ f_2$.



$\begin{cases} a_{11}x_1+\dots+a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1+\dots+a_{mn}x_n = b_m \end{cases}$

$A = (a_{ij}) \in M_{m\times n}(F)$ $\quad \longleftrightarrow \quad Ax = b$.

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in M_{m\times 1}(F)$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M_{n\times 1}(F)$

$\begin{bmatrix} & A & \end{bmatrix} \cdot \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} = \begin{bmatrix} C_1 \end{bmatrix}$

T US CONSTRUCT A LINEAR MAP ASSOCIATED TO THE SYSTEM. $T: F^n \longrightarrow F^m$.

$T(e_1) = T(1,0,\dots,0) = C_1 = (a_{11}, a_{21}, \dots, a_{m1})$

$T(e_2) = T(0,1,0,\dots,0) = C_2 = (a_{12}, a_{22}, \dots, a_{m2})$

$\vdots$

$T(e_n) = T(0,\dots,0,1) = C_n = (a_{1n}, \dots, a_{mn})$

$T(x_1, x_2, \dots, x_n) = T(x_1 \cdot e_1 + x_2 \cdot e_2 + \dots + x_n \cdot e_n) =$

$= x_1 T(e_1) + x_2 T(e_2) + \dots + x_n T(e_n)$

$= (a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n, a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n, \dots, a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n)$

$T(x) = Ax$

Def: let $T \in Hom_F(V,W)$.

1) $\operatorname{Ker} T = \operatorname{Null} T =$ KERNEL of T = NULL SPACE of $T = \{v \in V: T(v) = 0\} \subseteq V$.

2) $\operatorname{Im} T = \operatorname{Range} T =$ Image of T = RANGE of T. $= \{T(v), v \in V\} \subseteq W$.

Remark:

If $T: F^n \to F^m$ is the LM associated to a system of Linear Equations:

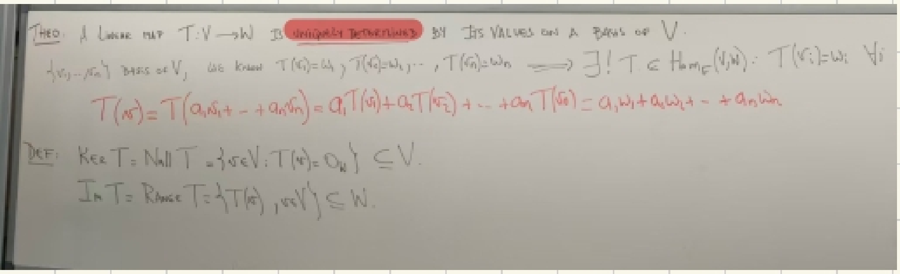
$T(x_1 \dots x_n) = (a_{11}x_1 + \dots + a_{1n}x_n \dots a_{m1}x_1 + \dots + a_{mn}x_n)$ THEN:

$\ker T = \{(x_1 \dots x_n) : \begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = 0 \end{cases}$ = Solutions of the Homogeneous system $Ax=0$.

$(b_1 \dots b_m) \in \operatorname{Im} T \Leftrightarrow (b_1 \dots b_n) = T(x_1 \dots x_n) = (a_{11}x_1 + \dots + a_{11}x_n, \dots, a_{m1}x_1 + \dots + a_{mn}x_n). \Leftrightarrow Ax = b.$

It is consistent.

4.28. ClASS 15.



THEO: A LINEAR MAP $T: V \to W$ IS UNIQUELY DETERMINED BY ITS VALUES ON A BASIS OF V.

$\{v_1 \dots v_n\}$ BASIS OF V, WE KNOW $T(v_1) = w_1, T(v_2) = w_2, \dots, T(v_n) = w_n \Longrightarrow \exists! T \in \operatorname{Hom}_F(V,W) : T(v_i) = w_i \ \forall i$

$T(v) = T(a_1v_1 + \dots + a_nv_n) = a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n) = a_1w_1 + a_2w_2 + \dots + a_nw_n$

DEF: $\operatorname{Ker} T = \operatorname{Null} T = \{v \in V : T(v) = 0_W\} \subseteq V.$

$\operatorname{Im} T = \operatorname{Range} T = \{T(v), v \in V\} \subseteq W.$

Example:

1) find $T \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^3)$ st $T(1,0) = (2.1,3)$. $T(0,1) = (-3,1,0)$.

$T(x,y) = T(x(1,0) + y(0,1)) = xT(1,0) + yT(0,1). = x(2,1,3) + y(-3,1,0) = (2x-3y, x+y, 3x).$

$\Rightarrow \begin{cases} 2x-3y = b_1 \\ x+y = b_2 \\ 3x = b_3 \end{cases} \Leftrightarrow \underset{A}{\begin{pmatrix} 2 & -3 \\ 1 & 1 \\ 3 & 0 \end{pmatrix}} \cdot \underset{x}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{b}{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}.$ $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \operatorname{Im} T \Leftrightarrow \exists (x,y) \in \mathbb{R}^2$. st $T(x,y) = (b_1 b_2 b_3)$.

$\Leftrightarrow (x,y)$ is a solution of the system $Ax = b$.

$\ker T = \{(x,y) : T(x,y) = (0,0,0)\} = \{(x,y) : \begin{cases} 2x-3y=0 \\ x+y=0 \\ 3x=0 \end{cases}\}$ = set of solution on $Ax=0$.

$= \{(0,0)\} \Rightarrow \dim \ker T = 0.$ is a subspace of $\mathbb{R}^2$.

$\operatorname{Im} T = \{T(x,y) : x,y \in \mathbb{R}^2\} = \{(2x-3y, x+y, 3x) : x,y \in \mathbb{R}\}.$

$(2x-3y, x+y, 3x) = x(2,1,3) + y(-3,1,0). \Rightarrow \langle (2,1,3), (-3,1,0) \rangle$ LI is subspace

$\Rightarrow \dim \operatorname{Im} T = 2.$

$\dim \mathbb{R}^2 = 2 = \dim \ker T + \dim \operatorname{Im} T.$

Proposition:

1) $\ker T$ is a subspace of V.

2) $\operatorname{Im} T$ is a subspace of W.

Proof:

1) $(0,0) \in \ker T$. since $T(0,0) = (0,0,0)$

- let $(x,y), (x',y') \in \ker T. \Rightarrow T((x,y) + (x',y')) = (0,0,0) + (0,0,0) = (0,0,0)$
- $\lambda \in \mathbb{R}$. $(x,y) \in \ker T \Rightarrow \lambda \cdot T(x,y) \in \ker T$.

$T(\lambda(x,y)) = \lambda \cdot T(x,y) = \lambda \cdot (0,0,0) = (0,0,0) \Rightarrow \ker T$ is a subspace

2). $\operatorname{Im} T = \{T(x,y) : (x,y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^3.$

- $(0,0,0) = T(0,0) \in \operatorname{Im} T$
- $(w_1, w_2, w_3), (u_1, u_2, u_3) \in \operatorname{Im} T. \overset{?}{\Longrightarrow} (w_1, w_2, w_3) + (u_1 + u_2 + u_3) \in \operatorname{Im} T$?

$(w_1, w_2, w_3) + (u_1, u_2, u_3) = T(x, y) + T(x', y') = T(x+x', y+y') \in ImT$.

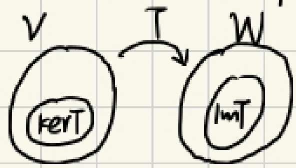
- $\lambda \in R$, $(w_1, w_2, w_3) \in ImT \Rightarrow \lambda(w_1, w_2, w_3) = \lambda \cdot T(x, y) = T(\lambda x, \lambda y) \in ImT$. ☒

LEMMA: if $\{v_1, v_2, \cdots v_n\}$ is a basis of V. $T \in Hom_F(V, W)$. then $ImT = span\{T(v_1), T(v_2), \cdots T(v_n)\}$.

Proof:

We have to prove that $\{T(v), v \in V\} = ImT = span(T(v_1), \dots T(v_n)) =$ set of L.C. of $T(v_1) \cdots T(v_n)$.
$v \in V$, $\{v_1 \dots v_n\}$ is a basis of V. $\Rightarrow v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. $T(v) = T(a_1v_1 + \cdots + a_nv_n)$
$\Rightarrow T(v) = a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n)$. $\subseteq$, $\supseteq$. ☒

THEO: let $T \in Hom_F(V, W)$. $dim_F V < \infty$ then $dim_F V = dim_F kerT + dim_F ImT$.



Proof:

We consider a basis of $kerT$. $\{v_1, v_2, \dots, v_r\} \Rightarrow$ it is LI in V. We extend it to a basis of V: $\{v_1, v_2, \cdots v_r, v_{r+1} \dots v_n\}$. $n = dim V$. $r = dim kerT \Rightarrow ImT$ is spanning by the set $\{T(v_1)=0 \dots T(v_r)=0 \dots T(v_n)\}$.
$ImT = span(T(v_1) \dots T(v_r), T(v_{r+1}) \dots T(v_n)). = span(T(v_{r+1}) \dots T(v_n))$.
$\Rightarrow \langle T(v_{r+1}) \dots T(v_n) \rangle$ is a spanning set for ImT. Now we have to prove it is LI in W.
let $0 = b_{r+1}T(v_{r+1}) + b_{r+2}T(v_{r+2}) \dots + b_nT(v_n)$. $\Rightarrow 0 = T(b_{r+1}v_{r+1} + \cdots + b_nv_n) \Rightarrow b_{r+1}v_{r+1} \cdots + b_nv_n \in kerT$. And $\{v_1, \dots v_r\}$ is a basis of $kerT$. $\Rightarrow b_{r+1}v_{r+1} + \cdots + b_nv_n = c_1v_1 + c_2v_2 + \cdots + c_rv_r \Rightarrow (-c_1)v_1 + (-c_2)v_2 + \cdots + (-c_r)v_r + b_{r+1}v_{r+1} + \cdots + b_nv_n = 0$. Since $\{v_1 \dots v_n\}$ is LI $\Rightarrow c_1 = c_2 = c_3 \cdots = c_r = 0$. $b_{r+1} = b_{r+2} = \cdots = b_n = 0$. $\Rightarrow$ it is LI in W. $\Rightarrow dim ImT = dim V - dim kerT$. ☒

Example

$T: R^2 \rightarrow R[x]$ the linear map determined by $T(1,0) = 1+x$. $T(0,1) = x - x^3$.
$(a, b) \rightarrow T(a, b) = aT(1,0) + bT(0,1). = a(1+x) + b(x - x^3) = a + (a+b)x - bx^3$.
$kerT = \{(a,b) : T(a,b) = a + (a+b)x - bx^3 = 0\} = \{(0,0)\} \Rightarrow dim kerT = 0$. $\Rightarrow dim ImT = dim R^2 - dim kerT = 2 - 0 = 2$.
Basis for ImT: $ImT = span(T(1,0), T(0,1)) = span(1+x, x+x^3)$. $\Rightarrow \{1+x, x+x^3\}$ is a basis of ImT.

Def:

let $T \in Hom_F(V, W)$.

1) T is a MONOMORPHISM $\Leftrightarrow$ T is an inj. M.
2) T is an EPIMORPHISM $\Leftrightarrow$ T is a Surj. E.
3) T is an ISOMORPHISM $\Leftrightarrow$ T is a Bij. I.

Proposition

1) T is M $\Leftrightarrow kerT = \{0\}$.
2) T is E $\Leftrightarrow ImT = W$.
3) T is I $\Leftrightarrow kerT = \{0\} \wedge ImT = W. \Leftrightarrow \exists T^{-1}: W \rightarrow V$. $T^{-1} \in Hom_F(W, V)$: $T \circ T^{-1} = id_W$. $T^{-1} \circ T = id_V$.

Proof.

1) $\Leftarrow$) Assume $kerT = \{0\}$. prove that T is inj. let $T(v_1) = T(v_2) \Rightarrow 0 = T(v_1) - T(v_2) = T(v_1 - v_2)$. Since $v_1 - v_2 \in kerT = \{0\} \Rightarrow v_1 - v_2 = 0 \Rightarrow v_1 = v_2$

$\Rightarrow$). We know that $\{0\} \subseteq \ker T$ since $T(0)=0$. let $v \in \ker T \Rightarrow T(v)=0=T(0) \Rightarrow v=0$ (inj.).

2) By Definition: T surj. $\Leftrightarrow \operatorname{Im}T = \{T(v), v \in V\} = W$.

3) By (1)(2) = T bij. $\Leftrightarrow T$ inj and surj. $\Leftrightarrow \ker T = \{0\}$ and $\operatorname{Im}T = W$.

We will prove that T is I (T is LM, T is bij.) $\Longleftrightarrow T$ has an Inverse $T^{-1} \in \operatorname{Hom}_F(W,V)$.

Since T is Bij $\Leftrightarrow T$ has inverse. We just need to prove T is LM $\Leftarrow T$ has inverse.

let T admit an Inverse $T^{-1}: W \to V$. we have to prove that T^{-1} is LM $\Rightarrow T$ LM.

let $T(v)=w$. $T(v')=w'$. $\Rightarrow T^{-1}(w)=v$. $T^{-1}(w')=v'$.

$T^{-1}(w+w') = T^{-1}(T(v)+T(v')) = T^{-1}(T(v+v')) = T^{-1}\circ T(v+v') = v+v' = T^{-1}(w)+T^{-1}(w')$. ✓

$T^{-1}(\lambda w) = T^{-1}(\lambda T(v)) = T^{-1}(T(\lambda v)) = T^{-1}\circ T(\lambda v) = \lambda v = \lambda \cdot T^{-1}(w)$. ✓

$\Rightarrow$ is LM.

THEO: let $T \in \operatorname{Hom}_F(V,W)$, $\dim_F V = n$. $\dim_F W = m$.

1) If T is an Epimorphism $\Rightarrow n \geq m$. In particular: $n<m \Rightarrow \nexists T: V \to W$ E.

2) If T is a Monomorphism $\Rightarrow n \leq m$. $n>m \Rightarrow \nexists T: V \to W$ M.

3) If T is an Isomorphism $\Rightarrow n=m$. $n \neq m \Rightarrow \nexists T: V \to W$ I.

Proof: we use $n = \dim V = \dim\ker T + \dim \operatorname{Im}T$.

a). T is E $\Leftrightarrow \operatorname{Im}T = W \Rightarrow n = \dim\ker T + \dim W = \dim\ker + m \geq m$. $\Rightarrow n \geq m$.

b). T is M $\Leftrightarrow \ker T = \{0\} \Rightarrow n = 0 + \dim \operatorname{Im}T \leq \dim W = m$. $\Rightarrow n \leq m$.

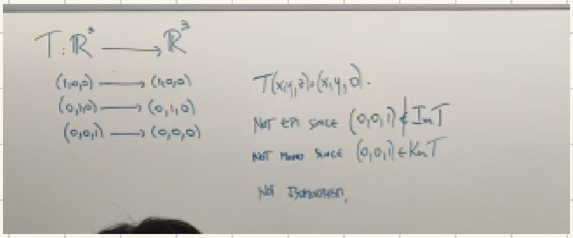
c) T is I $\Leftrightarrow T$ is E and M $\Rightarrow \operatorname{Im}T = W \wedge \ker T = \{0\}$. $\Rightarrow n = \dim\ker T + \dim \operatorname{Im}T = m$. $\Rightarrow n=m$.

Remark:

$n \geq m \not\Rightarrow T$ is E.

$n \leq m \not\Rightarrow T$ is M.

$n = m \not\Rightarrow T$ is I.



$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$(1,0,0) \longrightarrow (1,0,0)$

$(0,1,0) \longrightarrow (0,1,0)$

$(0,0,1) \longrightarrow (0,0,0)$

$T(x,y,z) = (x,y,0)$.

Not Epi since $(0,0,1) \notin \operatorname{Im}T$

Not Mono since $(0,0,1) \in \ker T$

Not Isomorphism.

4.30. CLASS 16.

$T: V \to W$ LM.

T is Monomorhism $\overset{Def}{\Longleftrightarrow}$ T is an injective function $\overset{Theo}{\Longleftrightarrow}$ $\ker T = \{v \in V: T(v)=0\} = \{0\}$.

T is Epimorphism $\overset{Def}{\Longleftrightarrow}$ T is a surjective function $\overset{Theo}{\Longleftrightarrow}$ $\operatorname{Im}T = W$.

T is Isomorphism $\overset{Def}{\Longleftrightarrow}$ T is a bijective function. $\overset{Theo}{\Longleftrightarrow}$ $\exists T^{-1}: W \to V$ a function is LM.

If $\langle v_1 \ldots v_n \rangle$ is a basis of V. then $\operatorname{Span}(T(v_1), T(v_2) \ldots T(v_n)) = \operatorname{Im}T$.

Theo: if $\dim V = n < \infty$, then $\dim V = \dim\ker T + \dim \operatorname{Im}T$. $\ker T \subseteq V$. $\operatorname{Im}T \subseteq W$.

Corollary: $\dim V = n$ $\dim W = m$.

1) T monomorphism $\Longrightarrow n \leq m$.

2) T epimorphism $\Rightarrow n \geq m$.

3) T isomorphism $\Rightarrow n = m$.

THEO: If $\dim V = n$, $B = \langle v_1, v_2 \ldots v_n \rangle \in V$ FAE (Following are equal).

a) B is a basis. b) B is a LI set. c) B is a spanning set of V.

Corollary: If $T \in Hom_F(V, W)$. $\dim V = \dim W = n$. then FAE:

a) T is an I.

b) T is a M.

c) T is an E.

Proof:

a) → b) ✓. T is I ⇒ T bij ⇒ T is inj ⇒ T is M.

b) → c): Assume T is monomophism ⇒ $\ker T = \{0\}$ ⇒ $\dim \ker T = 0$. Since $\dim V = \dim \ker T + \dim \operatorname{Im} T$.
⇒ $\dim V = \dim \operatorname{Im} T = n = \dim W$. Since $\operatorname{Im} T \subseteq W$ ⇒ $\operatorname{Im} T = W$. ⇒ T is E.

c) → a) Assume T is I. ⇒ $\operatorname{Im} T = W$. ⇒ $\dim \operatorname{Im} T = \dim W = n$. ⇒ $n = \dim V = \dim \operatorname{Im} T + \dim \ker T$.
⇒ $\dim \ker T = 0$ ⇒ $\ker T = \{0\}$ ⇒ T is M ⇒ T is I.

Remark: the previous Theo is not true for $\dim V = \dim W = \infty$.

Example:

1) $T: F[x] \longrightarrow F[x]$ $p(x) \longrightarrow p'(x)$. LM ✓. since $\begin{cases} T(p_1(x)+p_2(x)) = (p_1(x)+p_2(x))' = p_1'(x)+p_2'(x) \\ T(\lambda p(x)) = (\lambda p(x))' = \lambda \cdot p'(x) = \lambda T(p(x)). \end{cases}$

T is not mono: $T(\lambda) = 0$ $\forall \lambda \in F$.

T is Epi: $a_0 + a_1 x + a_2 x^2 = (a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3})' = T(a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3})$.

2) $T: F[x] \to F[x]$, $T(p(x)) = x \cdot p(x)$. LM. Since $\begin{cases} T(p_1(x)+p_2(x)) = x(p_1(x)+p_2(x)) = xp_1(x) - xp_2(x) = T(p_1(x)) + T(p_2(x)). \\ T(\lambda p(x)) = x \cdot (\lambda p(x)) = \lambda \cdot (x \cdot p(x)) = \lambda \cdot T(p(x)). \end{cases}$

T is Mono: $\ker T = \{a_0 + a_1 x + \cdots + a_n x^n : T(a_0 + a_1 x + \cdots + a_n x^n) = a_0 x + a_1 x^2 + \cdots + a_n x^{n+1} = 0\} = \{0\}$.

T is not Epi since $1 \neq x \cdot p(x)$. $1 \notin \operatorname{Im} T \Rightarrow \operatorname{Im} T \neq W$.

Example. $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x+y+z, x-z, x-y)$

a) T is LM.

b) T is an E.

$\operatorname{Im} T = \operatorname{span}(T(1,0,0), T(0,1,0), T(0,0,1)) = \operatorname{span}((1,1,1), (1,0,-1), (1,-1,0)) \subseteq \mathbb{R}^3$.

prove it is LI: $a(1,1,1) + b(1,0,-1) + c(1,-1,0) = (0,0,0) \Rightarrow a = b = c = 0 \Rightarrow$ it is LI.

$\dim \operatorname{Im} T = 3$, Since $\{(1,1,1), (1,0,-1), (1,-1,0)\}$ is a basis ⇒ $\operatorname{Im} T = \mathbb{R}^3$. ⇒ T is E.

c) CONCLUDE THET T IS I, and compute T^{-1}.

T^{-1} is LM.

$\left.\begin{array}{l} T(1,0,0) = (1,1,1) \\ T(0,1,0) = (1,0,-1) \\ T(0,0,1) = (1,-1,0) \end{array}\right\}$ basis of $\mathbb{R}^3$ ⇒ $\begin{array}{l} T^{-1}(1,1,1) = (1,0,0). \\ T^{-1}(1,0,-1) = (0,1,0). \\ T^{-1}(1,-1,0) = (0,0,1). \end{array}$

$T^{-1}(x,y,z) = T^{-1}(a(1,1,0) + b(1,0,-1) + c(1,-1,0)) = a(1,0,0) + b(0,1,0) + c(0,0,1). = (a,b,c)$.

$\Rightarrow \begin{cases} a+b+c = x \\ a-c = y \\ a-b = z \end{cases} \Rightarrow \begin{cases} a = \frac{x+y+z}{3} \\ b = \frac{x+y-2z}{3} \\ c = \frac{x-2y+z}{3} \end{cases} \Rightarrow (a,b,c) = \left(\frac{x+y+z}{3}, \frac{x+y-2z}{3}, \frac{x-2y+z}{3}\right)$.

How can we detect: Mono (using LI set) Epimo (using spanning set) Isomo (using basis).

THEO 1: $T: V \to W$ is a Mono ⟺ T transforms LI sets into LI sets:

For any $X \subseteq V$ X is LI, we take $T(X)$ is LI in W.

THEO 2: $T: V \to W$ is an Epimo $\Leftrightarrow$ it transforms spanning sets of V into spanning sets of W:
$\forall X \subseteq V$, $V = \text{span}(X) \Rightarrow \text{span}(T(X)) = W$.

THEO 3: $T \in \text{Hom}_F(V, W)$ then TFAE:
1) T is an Isomo.
2) T transforms any basis B of V into a basis $T(B)$ of W.
3) $\exists B$ basis of V st $T(B)$ is a basis of W.

Proof theo 1:

$\Rightarrow$) Assume T is a Mono. we will prove that $\{v_1 \dots v_n\}$ is LI $\Leftrightarrow \{T(v_1), T(v_2) \dots T(v_n)\}$ LI.
$a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n) = 0 \Longleftrightarrow T(a_1v_1 + \dots + a_nv_n) = 0 \Longleftrightarrow a_1v_1 + \dots + a_nv_n = 0$.
$\Rightarrow$) if $\{v_1, \dots v_n\}$ LI then $a_1v_1 + \dots + a_nv_n = 0 \Leftrightarrow a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n) = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$.
$\Leftarrow$) if $\{T(v_1) \dots T(v_n)\}$ LI then $a_1T(v_1) + \dots + a_nT(v_n) = 0 \Leftrightarrow a_1v_1 + a_2v_2 + \dots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$.
$\Leftarrow$) Assume that X LI in $V \Leftrightarrow T(X)$ LI in W. We will prove that T is monomorphism:
Assume T is not a Mono. $\exists v \neq 0$ st $T(v) = 0$. then $X = \{v\}$ is LI set, $T(X) = \{T(v) = 0\} = \{0\}$ is not LI. Contradiction. $\Rightarrow T$ is Mono ▨

Proof THEO 2.

$\Rightarrow$) Assume that T is Epi. we will prove $V = \text{span}(Y) \Rightarrow W = \text{span}(T(Y))$.
let $V = \text{Span}(v_1, \dots v_k)$: $v = a_1v_1 + a_2v_2 + \dots + a_kv_k \Rightarrow T(v) = a_1T(v_1) + a_2T(v_2) + \dots + a_kT(v_k)$. $\forall v \in V$
$\Rightarrow \text{span}(T(v_1), T(v_2), \dots T(v_k)) = \text{Im}T = W$ since T is Epi.
$\Leftarrow$) Assume that $V = \text{span}(Y) \Rightarrow W = \text{span}(T(Y))$ we will prove that T is Epi.
By Contradiction, T is not Epi then $\text{Im}T \subsetneq W$. Take $Y = V \Rightarrow V = \text{span}(V)$ but $\text{span}(T(V)) = \text{Im}T \subsetneq W$ Since $\text{span}(T(Y)) = W \Rightarrow$ Contradiction ▨

Proof THEO 3.

From the theo 1. theo 2. we know that
T is Iso $\Leftrightarrow$ for any basis B of V. $T(B)$ is LI. $T(B)$ is spanning set.
That is, $(1) \Leftrightarrow (2)$. $(2) \Leftrightarrow (3)$ is CLEAR: True for any basis $\Rightarrow$ true for one basis.
let's see $3) \Rightarrow 1)$:
Assume $\{v_1 \dots v_n\}$ is a basis of V. and $\{T(v_1), T(v_2) \dots T(v_n)\}$ is a basis of W.
T Epi: $W = \text{Im}T$: $w \in W$, $w = a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n) = T(a_1v_1 + \dots + a_nv_n) \in \text{Im}T \Rightarrow W \subseteq \text{Im}T$.
$\Rightarrow W = \text{Im}T$.
T Mono: $v \in \ker T$ $v = a_1v_1 + \dots + a_nv_n \Rightarrow 0 = T(v) = a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n) \Rightarrow a_1 = a_2 = \dots = a_n = 0$.
$\Rightarrow v = 0. \Rightarrow T$ is Mono. ▨

Proposition: $\dim V = n$, $\dim W = m$. if $n \leq m$, then $\exists T: V \to W$ which is Mono.
Proof: Take $\{v_1 \dots v_n\}$ a basis of V. take $\{w_1 \dots w_n\}$ a LI set of W.
define $T: V \to W$ as the LM. $T(v_i) = w_i$ $\forall i$ let $T(v) = 0 \Rightarrow v = a_1v_1 + \dots + a_nv_n \Rightarrow$
$0 = T(v) = a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n) = a_1w_1 + a_2w_2 + \dots + a_nw_n \Rightarrow a_1 = a_2 = \dots = a_n = 0 \Rightarrow v = 0 \Rightarrow T$ Mono.

Example:

Describe the $T \in \mathrm{Hom}_F(\mathbb{R}^3, \mathbb{R}^3)$: $T(1,0,0) = (1,1,1)$ $T(0,1,0) = (0,1,1)$. $T(0,0,1) = (0,0,1)$.

$\Rightarrow$ 3 LI vectors $\Rightarrow$ basis. $T(B) = B'$. B, B' basis $\Rightarrow T$ is Iso.

$T(x,y,z) = T(x(1,0,0) + y(0,1,0) + z(0,0,1)) = x(1,1,1) + y(0,1,1) + z(0,0,1) = (x, x+y, x+y+z)$.

Now describe T^{-1}. $T^{-1}(1,1,1) = (1,0,0)$. $T^{-1}(0,1,1) = (0,1,0)$. $T^{-1}(0,0,1) = (0,0,1)$.

$\Rightarrow T^{-1}(x,y,z) = T^{-1}(x(1,1,1) + (y-x)(0,1,1) + (z-y)(0,0,1)) = (x, y-x, z-y)$.

$T \circ T^{-1}(x,y,z) = T(x, y-x, z-y) = (x, x+y-x, x+y-x+z-y) = (x,y,z)$.

$T^{-1} \circ T(x,y,z) = T^{-1}(x, x+y, x+y+z) = (x, x+y-x, x+y+z-x-y) = (x,y,z)$.

5.7. CLASS 17.

AN APPLICATION OF LINEAR MAPS TO SYSTEMS OF LINEAR EQUATIONS.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Longleftrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$AX = b$ $A \in M_{m\times n}(F)$ $b \in M_{m\times 1}(F)$.

$\longleftrightarrow T: F^n \longrightarrow F^m$:

$T(x_1 \ldots x_n) = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, a_{21}x_1 + \cdots, \ldots, a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n)$.

$$A \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

Def: $A \in M_{m\times n}(F)$.

row rank $= \dim_F \mathrm{span}(R_1, R_2 \ldots R_m)$.

Column rank $= \dim_F \mathrm{span}(C_1, C_2, \ldots C_n)$.

Example:

$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 4 \end{pmatrix}$ row rank $= \dim_F \mathrm{span}((1,2,-1,0), (0,1,2,4)) = 2$.

column rank $= \dim_F \mathrm{span}((1,0), (2,1), (-1,2), (0,4)) = 2$.

THEO: row-rank(A) = column-rank(A).

Proof:

let A' be a Row Reduced Echelon Form of A. $A \longrightarrow A' = \begin{bmatrix} 0 \cdots 0 & a_{1s_1} & 0 & 0 & \cdots & 0 \\ 0 \cdots 0 & 0 & \cdots & a_{2s_2} & \cdots & 0 \\ 0 \cdots 0 & \cdots & \cdots & \cdots & a_{3s_3} \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \; 0 \end{bmatrix}$ for $a_{is_i} = 1$.

then row-rank$(A) = \dim_F((0,0 \cdots a_{1s_1} \cdots 0), (0,0 \cdots a_{2s_2} \cdots 0) \ldots (0,0 \cdots a_{is_i} \cdots 0) \ldots (0,0 \ldots a_{ks_k} \ldots 0), (0,0 \cdots 0), \ldots) = k$.

column-rank$(A) = \dim_F(C_{s_1}, \underbrace{C_{s_1+1}, \ldots}_{LD} C_{s_2}, \underbrace{C_{s_2+1}, \ldots}_{LD}, C_{s_k}, \underbrace{C_{s_k+1}, \ldots}_{LD.} C_n) = k$.

Now we have prove rank of A'. We need to prove (I) Row-rank(A) = Row-rank(A').

(II) Colum-rank(A) = Column-rank(A').

Proof:

(I) We will prove that the Row-rank does not change if we apply ELEMENTARY ROW-OPERATION

$R_i \longleftrightarrow R_j$: $\mathrm{span}(R_1 \ldots R_i \ldots R_j \ldots R_n) = \mathrm{span}(R_1 \ldots R_j \ldots R_i \ldots R_n)$. ✓

$a_1R_1 + a_2R_2 + \cdots + a_iR_i + \cdots + a_jR_j + \cdots + a_nR_n = a_1R_1 + \cdots + a_jR_j + \cdots + a_iR_i + \cdots + a_nR_n$

$R_i \longleftrightarrow \lambda R_i$. $\lambda \neq 0$: $\text{span}(R_1 \dots R_i \dots R_n) = \text{span}(R_1 \cdots \lambda R_i \dots R_n)$.

$a_1R_1 + a_2R_2 + \cdots + a_iR_i + \cdots + a_nR_n = b_1R_1 + \cdots + b_i\lambda R_i + \cdots + b_nR_n$.

let $a_k = b_k \ \forall k \in [n] \setminus \{i\}$. $a_i = b_i\lambda$.

$R_i \longleftrightarrow R_i + \lambda R_j$. $\lambda \neq 0$: $\text{span}(R_1 \dots R_i \dots R_j \dots R_n) = \text{span}(R_1 \cdots R_i + \lambda R_j \dots R_j \dots R_n)$.

$a_1R_1 + \cdots + a_iR_i + \cdots + a_jR_j + \cdots + a_nR_n = b_1R_1 + \cdots + b_iR_i + b_i\lambda R_j + \cdots + b_jR_j + \cdots + b_nR_n$.

$\Rightarrow$ let $a_k = b_k \ \forall k \in [n] \setminus \{j\}$. $a_j = b_i\lambda + b_j$. ☒

(II). $T: F^n \to F^m$, $T(x^T) = (Ax)^T$. $T': F^n \to F^m$, $T(x^T) = (A'x)^T$. $X = (e_1 e_2 \dots e_n)$.

Column-rank$(A) = \dim_F \text{span}(C_1 C_2 \dots C_n) = \dim_F \text{span}(T(e_1), T(e_2), \cdots T(e_n)). = \dim_F \operatorname{Im} T$.

$= \dim F^n - \dim \ker T$.

Column-rank$(A') = \dim_F \text{span}(C_1' C_2' \dots C_n') = \dim_F \text{span}(T'(e_1)\ T'(e_2) \cdots T'(e_n)) = \dim \operatorname{Im} T' = \dim F^n - \dim \ker T'$.

$\ker T = \{x^T : T(x^T) = 0\} = \{x : Ax = 0\}$.

$\ker T' = \{x^T : T'(x^T) = 0\} = \{x : A'x = 0\}$ Since $A' = E_1E_2 \cdots E_kA$. Then $Ax = 0 \Leftrightarrow A'x = 0$.

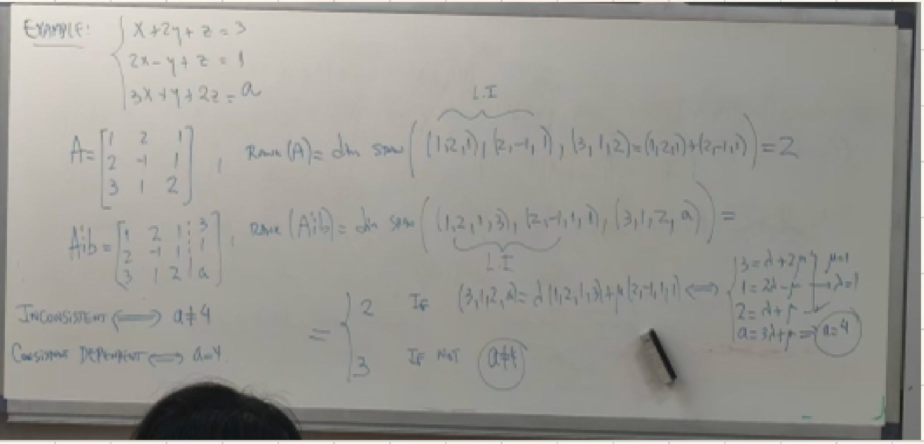
So $\ker T = \ker T'$. $\Rightarrow$ both equal ☒

THEO: let $A \in M_{m\times n}(F)$, $Ax = b$. a system of m linear equations with n unknowns.

1) The system is **INCONSISTENT** $\Leftrightarrow$ $\text{rank}A < \text{rank}(A|b)$. $A \in M_{m\times n}(F)$.

2) is **CONSISTENT INDEPENDENT** $\Leftrightarrow$ $\text{rank}A = \text{rank}(A|b) = n$ $(A|b) \in M_{m\times(n+1)}(F)$.

3) is **CONSISTENT DEPENDENT** $\Leftrightarrow$ $\text{rank}A = \text{rank}(A|b) < n$.



EXAMPLE: $\begin{cases} x + 2y + z = 3 \\ 2x - y + z = 1 \\ 3x + y + 2z = a \end{cases}$

$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $\text{Rank}(A) = \dim \text{span}\left(\overbrace{(1,2,1), (2,-1,1)}^{L.I}, (3,1,2) = (1,2,1) + (2,-1,1)\right) = 2$

$A|b = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & 2 & a \end{array}\right]$, $\text{Rank}(A|b) = \dim \text{span}\left(\underbrace{(1,2,1,3), (2,-1,1,1)}_{L.I}, (3,1,2,a)\right) =$

$= \begin{cases} 2 & \text{If } (3,1,2,a) = \lambda(1,2,1,3) + \mu(2,-1,1,1) \Longleftrightarrow \begin{cases} 3 = \lambda + 2\mu \\ 1 = 2\lambda - \mu \\ 2 = \lambda + \mu \\ a = 3\lambda + \mu \end{cases} \begin{matrix} \mu = 1 \\ \lambda = 1 \end{matrix} \Rightarrow \boxed{a = 4} \\ 3 & \text{If NOT } \boxed{a \neq 4} \end{cases}$

INCONSISTENT $\Longleftrightarrow a \neq 4$

Consistent DEPENDENT $\Longleftrightarrow a = 4$

Proof of the theo: $Ax = b \leftrightarrow T: F^n \to F^m$, $T(x^T) = (Ax)^T$.

$\text{Rank}A = \dim \text{span}(C_1 C_2 \dots C_n) = \dim \text{span}(T(e_1), T(e_2) \dots T(e_n)) \leq \dim \text{span}(T(e_1) \dots T(e_n), b^T) = \dim\text{span}(C_1 C_2 \cdots C_n, b$

$= \text{Rank}(A|b)$.

We have that $\text{span}(T(e_1) \dots T(e_n)) = \text{span}(T(e_1), T(e_2) \cdots T(e_n), b^T) \Leftrightarrow b^T \in (T(e_1) \dots T(e_n)) = \operatorname{Im} T \Leftrightarrow \exists z \in F^n : T(z) = b^T$

$\Leftrightarrow$ the system has a solution $z \Leftrightarrow$ Consistent.

The system is Dependent $\Leftrightarrow \exists z_1, z_2$ st $z_1 \neq z_2$: $Az_1 = b$, $Az_2 = b$. $\Leftrightarrow \exists w = z_1 - z_2 \neq 0$: $Aw = A(z_1 - z_2) = 0$.

$\Leftrightarrow \exists w^T \neq 0 \in F^n$: $T(w^T) = 0$. $\Leftrightarrow$ T is not Mono $\Leftrightarrow \dim\ker T \geq 1$. and $\dim F^n = \dim \ker T + \dim \operatorname{Im} T$

$> \dim \operatorname{Im} T > \text{Rank}A$.

Independent $\Leftrightarrow$ T Mono $\Leftrightarrow \dim F^n = \dim\ker T + \dim \operatorname{Im} T \Leftrightarrow n = 0 + \text{rank}A$.

Corollary 1: $A \in M_{m \times n}(F)$ $Ax = 0$ the system is alway consistant.
And if $\operatorname{rank} A < n \Rightarrow$ Dependent. if $\operatorname{rank} A = n \Rightarrow$ Independent.

Proof: It is check that $\operatorname{rank} A = \operatorname{rank}(A|0)$. since $\operatorname{span}(C_1 \dots C_n) = \operatorname{span}(C_1 \dots C_n, 0)$.

Corollary 2: $A \in M_{m \times n}(F)$. $Ax = 0$. If $m < n \Rightarrow Ax = 0$ is Consistent dependent.

Proof: $T : F^n \to F^m$.

CI $\Leftrightarrow \operatorname{rank} A = n$. since $A \in M_{m \times n}(F) \Rightarrow n = \text{Column-rank} A = \dim \operatorname{span}(C_1 \cdots C_n)$ which is $\leq \dim F^m = m$. $\Rightarrow n \leq m$. Use the definition of Mono, and Epi to prove ...

INVERTIBLE MATRICES.

Def: $A \in M_{m \times n}(F)$.

1) We say that A has a LEFT INVERSE if $\exists B \in M_{n \times m}(F)$: $BA = Id_n$.
2) RIGHT INVERSE if $\exists C \in M_{n \times m}(F)$: $AC = Id_m$.
3) INVERSE if $\exists B \in M_{n \times m}(F)$: $BA = Id_n$. $AB = Id_m$.

Question: if we need $n = m$?

Example

$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix} \neq Id$

LAMMA 1: $A \in M_{m \times n}(F)$. If A has LEFT INVERSE $B \in M_{n \times m}$ and RIGHT INVERSE $C \in M_{n \times m}$.
then $B = C$.

Proof: $B = B \cdot Id_m = B \cdot (AC) = (BA)C = Id_n \cdot C = C$.

LAMMA 2: $A \in M_{n \times m}(F)$ and it has INVERSE, that it is unique.

Proof: Assume $B_1 A = Id_n$, $AB_1 = Id_m$. $\Rightarrow B_1$ is LEFT INVERSE, B_2 is RIGHT INVERSE.
$B_2 A = Id_n$, $AB_2 = Id_m$. From LAMMA 1. $\Rightarrow B_1 = B_2$.

THEO: $A \in M_n(F)$ FAE:

1) A has left inverse.
2) Rank $A = n$.
3) A is Row-Equiv. to Id.
4) A is a product of ELEMENTARY MATRICES.
5) A has inverses.

5.12. CLASS 18.

Def: $A \in M_{m\times n}(F)$.

1) A has LEFT INVERSE if $\exists B \in M_{n\times m}(F)$. $BA = Id_n$.

2) A has RIGHT INVERSE if $\exists C \in M_{n\times m}(F)$. $AC = Id_m$.

3) A has INVERSE or is INVERTIBLE, if $\exists B \in M_{m\times n}(F)$ $BA = Id_n$. $AB = Id_m$.

(We will prove that it implies $n=m$).

We have proved: 1) If $BA = Id_n$, $AC = Id_m \Rightarrow B = C$. 2) A has inverse $\Rightarrow$ it is unique

THEO 1: $A, B \in M_n(F)$.

1) A is INVERTIBLE $\Leftrightarrow A^{-1}$ is INVERTIBLE. $(A^{-1})^{-1} = A$.

2) $A.B$ INVERTIBLE $\Leftrightarrow A.B$ and $B.A$ are INVERTIBLE. $(AB)^{-1} = B^{-1}A^{-1}$. $(BA)^{-1} = A^{-1}B^{-1}$

Proof:

1) $A.A^{-1} = Id_n = A^{-1}A$.

2). Assume $A.B$ are INVERTIBLE:

$AB \cdot B^{-1}A^{-1} = Id = B^{-1} \cdot A^{-1} \cdot AB$ $\quad BA \cdot A^{-1}B^{-1} = Id = A^{-1}B^{-1}BA$.

THEO 2 $A \in M_n(F)$. Then TFAE:

1) A has left inverse.

2) RANK A $= n$.

3) A is Row Equivalent to Id_n.

4) A is a product of Row Elementary Matrices.

5) A has inverse.

Proof:

(5) $\Rightarrow$ (1) ✓.

(4) $\Rightarrow$ (5):

$A = E_1 \dots E_s$, E_i are Row Elementary Matrices. By theo 1. Using induction

$\Rightarrow A^{-1} = E_s^{-1} E_{s-1}^{-1} \dots E_2^{-1} E_1^{-1}$ ✓.

(3) $\Rightarrow$ (4):

$A \sim Id$ (Row Equivalent) $\Rightarrow \exists E_1, E_2, \dots E_s$ Row Elementary Matrices st $E_s \dots E_2 E_1 A = Id$

$\Rightarrow E_s^{-1}(E_s \dots E_1 A) = E_s^{-1} \cdot Id = E_s^{-1} \Rightarrow E_{s-1} \dots E_1 A = E_s^{-1} \Rightarrow A = E_1^{-1} \dots E_s^{-1}$ Since E_i^{-1} is also a Row Elementary M ✓

(1) $\Rightarrow$ (2):

Assume A has left inverse $\Rightarrow \exists B \in M_n(F)$: $BA = Id$. Consider $Ax = 0$: we know $\{x : Ax = 0\} = \{0\} \Leftrightarrow$ $Ax = 0$ CONSISTENT INDEPENDENT $\Leftrightarrow$ RANK A $= n$.

Since $T: F^n \to F^n$. $T(x^T) = (Ax)^T$, $Ax = 0$ Consistent Independent $\Leftrightarrow \ker T = \{x^T : T(x^T) = 0\} = \{x^T : (Ax)^T = 0\}$ $= \{x^T : Ax = 0\} = \{0\}$. $\Rightarrow \ker T = \{0\} \Rightarrow \dim \operatorname{Im} T = \dim V = n$. Now $\operatorname{Im} T = \operatorname{span}(T(e_1), \dots T(e_n)) = \operatorname{span}((Ae_1)^T \dots (Ae_n)^T)$ $= \operatorname{span}(c_1^T, \dots c_n^T) \Rightarrow \dim \operatorname{Im} T = \dim \operatorname{span}(c_1, c_2 \dots c_n) =$ Column rank $A =$ rank A.

We have proven: $\{x : Ax = 0\} = \{0\} \Rightarrow$ RANK $= n$. If $BA = Id \wedge Ax = 0 \Rightarrow B(Ax) = B \cdot 0$

$BAx = Id x \Rightarrow x = 0. \Rightarrow$ RANK $= n$.

(2) $\Rightarrow$ (3) :

Assume RANK = n. Then the ROW REDUCED ECHELON FORM of A

$$A \to \underset{\substack{\uparrow \\ \text{row operation}}}{\cdots} \to \tilde{A} = \begin{bmatrix} 0 \cdots 0\, a_{1s_1} \cdots 0\ 0 \\ 0 \cdots 0\ 0 \cdots a_{2s_2} \cdots 0 \\ \vdots \\ 0\ 0\ 0\ 0 \cdots \cdots a_{ks_k} \\ 0\ 0 \cdots \cdots 0 \end{bmatrix} \overset{?}{=} Id.$$

since $n = \text{RANK}A = \text{Row RANK}A = \text{Row RANK}\tilde{A} = k$

$$\Rightarrow \tilde{A} = \begin{bmatrix} 0 \cdots a_{1s_1} \cdots 0 \\ 0 \cdots \cdots a_{2s_2} \cdots 0 \\ \vdots \\ 0 \cdots \cdots \cdots a_{ns_n} \end{bmatrix} \to n \text{ Column.} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix} = Id \Rightarrow n = k. \checkmark$$

Remark: If $A \in M_{m\times n}(F)$ has left and right inverse, then $n = m$ and A is INVERTIBLE.

Proof: $\exists B \in M_{n\times m}(F), \exists C \in M_{n\times m}(F): BA = Id_n, AC = Id_m \Rightarrow B = B\,Id_m = B(AC) = (BA)C = Id_n C = C.$

$$\Rightarrow \left\{ \begin{array}{l} BA = Id_n \Rightarrow \text{rank}A = n \Rightarrow n = \text{rank}A = \text{row rank}A \leq m \\ AB = Id_m \Rightarrow \text{rank}B = m \Rightarrow m = \text{rank}B = \text{row rank}B \leq n \end{array} \right\} \Rightarrow n = m.$$

CONNECT SYSTEMS OF LINEAR EQUATIONS WITH LINEAR MAPS.

$Ax = b \longrightarrow T: F^n \to F^m, \ T(x_1 \dots x_n) = (a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n, \dots a_{1m}x_m + a_{2m}x_m \dots + a_{nm}x_m).$

$T: F^n \to F^m$, T is uniquely determined by $T(e_1), T(e_2) \dots T(e_n) \in F^m$.

$Ax = b \leftrightsquigarrow T(e_1) = (a_{11}, a_{21} \dots a_{m1})$

$T(e_2) = (a_{12}, a_{22} \dots a_{m2}). \quad \Rightarrow T(x_1, x_2 \dots x_n) = x_1 T(e_1) + \dots + x_n T(e_n).$

$\vdots$

$T(e_n) = (a_{1n}, a_{2n}, \dots a_{mn}).$

Now, connect LINEAR MAPS with MATRICES

Def: $T: V \to W$, LM between two F-vector spaces

let B_1 basis of V. B_2 basis of W, $B_1 = \{v_1 \dots v_n\}$. $B_2 = \{w_1 \dots w_m\}$.

The matrix of T with respect to B_1 and B_2 is defined by:

$$[T]_{B_1B_2} = M(T, B_1, B_2) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in M_{m\times n}(F) \text{ st } T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_m = \sum_{i=1}^{m} a_{ij}w_i.$$

$$[T]_{B_1B_2} = \left[[T(v_1)]_{B_2} \cdots [T(v_n)]_{B_2} \right]$$

Example: $T: R^2 \to R^3$, $T(x, y) = (2x+y, x, 3y+x)$.

$\mathcal{B}_2 = \{(1,0), (0,1)\}$ $\mathcal{B}_3 = \{(1,0,0), (0,1,0), (0,0,1)\}$. $B_1 = \{(1,0), (1,1)\}$. $B_2 = \{(1,0,0), (0,1,1), (0,-1,1)\}$

$T(1,0) = (2,1,1) = 2(1,0,0) + 1(0,1,0) + 1(0,0,1).$

$T(0,1) = (1,0,3) = 1(1,0,0) + 0(0,1,0) + 3(0,0,1).$

$$\Rightarrow [T]_{\mathcal{B}_2\mathcal{B}_3} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}$$

$T(1,0) = (2,1,1) = 2(1,0,0) + 1(0,1,1) + 0(0,-1,1).$

$T(1,1) = (3,1,4) = 3(1,0,0) + 1/2(0,1,1) + 3/2(0,-1,1)$

$$\Rightarrow [T]_{B_1B_2} = \begin{bmatrix} 2 & 3 \\ 1 & 1/2 \\ 0 & 3/2 \end{bmatrix}$$

if $\mathcal{B}_2 = \{(0,1), (1,0)\}$: $T(0,1) \to 1$ column. $T(1,0) \to 2$ column. $\Rightarrow [T]_{\mathcal{B}_2\mathcal{B}_3} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}.$

THEO:

1) $T, T': V \to W$, B_1 basis of V. B_2 basis of W.

$$[T + T']_{B_1B_2} = [T]_{B_1B_2} + [T']_{B_1B_2}$$

2) $\lambda \in F$, $T: V \to W$, B_1 basis of V. B_2 basis of W.

$$[\lambda T]_{B_1B_2} = \lambda [T]_{B_1B_2}$$

3) $T: V \to W$, $T': W \to U$, B_1 basis of V. B_2 basis of W, B_3 basis of U

$$[T' \circ T]_{B_1B_3} = [T']_{B_2B_3}[T]_{B_1B_2}.$$

Proof:

1) $B_1 = \{v_1 \dots v_n\}$, $B_2 = \{w_1 \dots w_m\}$.

$[T+T']_{B_1B_2} = \left[[(T+T')(v_1)]_{B_2} \,\vdots\, [(T+T')(v_2)]_{B_2} \,\vdots\, \cdots \,\vdots\, [(T+T')(v_n)]_{B_2} \right]$.

$(T+T')(v_j) = T(v_j) + T'(v_j) = \Sigma a_{ij} w_i + \Sigma a'_{ij} w_i = \Sigma (a_{ij} + a'_{ij}) w_i$. where $[T]_{B_1B_2} = [a_{ij}]$ $[T']_{B_1B_2} = [a'_{ij}]$.

$\Rightarrow [T+T']_{B_1B_2} = [a_{ij} + a'_{ij}] = [a_{ij}] + [a'_{ij}] = [T]_{B_1B_2} + [T']_{B_1B_2}$.

2) $[\lambda T]_{B_1B_2} = \left[[(\lambda T)(v_1)]_{B_2} \,\vdots\, \cdots \,\vdots\, [(\lambda T)(v_n)]_{B_2} \right]$

$(\lambda T)(v_{ij}) = \lambda \cdot T(v_{ij}) = \lambda \cdot \sum_{i=1}^{m} a_{ij} w_i = \sum_{i=1}^{m} (\lambda a_{ij}) w_i$. $\Rightarrow$ $[\lambda T]_{B_1B_2} = [\lambda a_{ij}] = \lambda [a_{ij}] = \lambda \cdot [T]_{B_1B_2}$.

3). $V \xrightarrow{T} W \xrightarrow{T'} U$. $B_3 = \{u_1 \dots u_k\}$.

$[T]_{B_1B_2} = [a_{ij}] \Leftrightarrow T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$. $\forall j \in 1, 2, \dots n$

$[T']_{B_2B_3} = [b_{ij}] \Leftrightarrow T'(w_t) = \sum_{s=1}^{k} b_{st} u_s$ $\forall t \in 1, 2 \dots m$.

$(T' \circ T)(v_j) = T'(T(v_j)) = T'(\sum_{i=1}^{m} a_{ij} w_i) = \sum_{i=1}^{m} a_{ij} T'(w_i) = \sum_{i=1}^{m} a_{ij} \sum_{s=1}^{k} b_{si} u_s$. $= \sum_{i=1}^{m} \sum_{s=1}^{k} b_{si} a_{ij} u_s$. $= \sum_{s=1}^{k} (\sum_{i=1}^{m} b_{si} a_{ij}) u_s$.

$\Rightarrow [T' \circ T]_{B_1B_3} = [(\sum_{i=1}^{m} b_{si} a_{ij})_{sj}] = [b_{st}][a_{ij}] = [T']_{B_2B_3}[T]_{B_1B_2}$.

Example.

$T(x,y) = (x+y, x-y)$ $T'(x,y) = (2x, x-y, 3y)$.

$(T' \circ T)(x,y) = T'(T(x,y)) = T'(x+y, x-y) = (2(x+y), x+y-(x-y), 3(x-y)) = (2x+2y, 2y, 3x-3y)$.

$\Rightarrow [T' \circ T]_{\varepsilon_2\varepsilon_3} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 3 & -3 \end{bmatrix}$ since $(T' \circ T)(1,0) = (2,0,3) = 2(1,0,0) + 0(0,1,0) + 3(0,0,1)$

$(T' \circ T)(0,1) = (2,2,-3) = 2(1,0,0) + 2(0,1,0) + (-3)(0,0,1)$.

Since $[T']_{\varepsilon_2\varepsilon_3} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}$ $[T]_{\varepsilon_2\varepsilon_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\Rightarrow [T']_{\varepsilon_2\varepsilon_3}[T]_{\varepsilon_2\varepsilon_2} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 3 & -3 \end{bmatrix}$ ✓.

5.14. CLASS 19.

Recall the last class:

$T: V \to W$ $B_1 = \{v_1 \dots v_n\}$. $B_2 = \{w_1 \dots w_n\}$ B_1 basis of V. B_2 basis of W.

then $[T]_{B_1B_2} = M(T, B_1, B_2) = \begin{bmatrix} a_{11} \cdots a_{1j} \cdots a_{1n} \\ \vdots \\ a_{m1} \cdots a_{mj} \cdots a_{mn} \end{bmatrix}$ $\to T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_m$.

$T': \overset{B_1}{V} \longrightarrow \overset{B_2}{W}$. $T+T': V \to W$. $\lambda T: V \to W$. $T'': \overset{B_2}{W} \to \overset{B_3}{U}$.

$[T+T']_{B_1B_2} = [T]_{B_1B_2} + [T']_{B_1B_2}$.

$[\lambda T]_{B_1B_2} = \lambda [T]_{B_1B_2}$

$[T'' \circ T]_{B_1B_3} = [T'']_{B_2B_3} \cdot [T]_{B_1B_2}$.

Example. $T, T': \mathbb{R}^2 \to \mathbb{R}^2$, ε_2 both basis. $T(x,y) = (2x, x+y)$. $T'(x,y) = (x-y, x+2y)$. check it!

$\Rightarrow (T+T')(x,y) = T(x,y) + T'(x,y) = (2x, x+y) + (x-y, x+2y) = (3x-y, 2x+3y)$. $\Rightarrow [T+T']_{\varepsilon_2\varepsilon_2} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}$.

$\Rightarrow (3T)(x,y) = 3T(x,y) = 3(2x, x+y) = (6x, 3x+3y)$. $\Rightarrow [3T]_{\varepsilon_2\varepsilon_2} = \begin{bmatrix} 6 & 0 \\ 3 & 3 \end{bmatrix}$

$[T]_{B_2B_2} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ $T(1,0) = (2,1) = 2(1,0) + 1(0,1)$.

$T(0,1) = (0,1) = 0(1,0) + 1(0,1)$. $\Rightarrow [T]_{B_2B_2} + [T']_{B_2B_2} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}$

$[T']_{B_2B_2} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$ $T'(1,0) = (1,1) = 1(1,0) + 1(0,1)$.

$T'(0,1) = (-1,2) = -1(1,0) + 2(0,1)$ $\Rightarrow \quad = [T+T']_{B_2B_2}$

and $3 \cdot [T]_{B_2B_2} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \cdot 3 = \begin{bmatrix} 6 & 0 \\ 3 & 3 \end{bmatrix} = [3T]_{B_2B_2}$.

We have proved : $\dim V < \infty$.

1) $T: V \to W$ Iso $\overset{\text{Def}}{\Leftrightarrow}$ T is a Linear Map, bijective function $\Leftrightarrow \exists T^{-1}: W \to V$ Linear Map st $T \circ T^{-1} = Id_W$. $T^{-1} \circ T = Id_V$

2) $T: V \to W$. Iso $\Rightarrow \dim V = \dim W$.

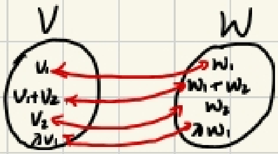
3) If $\dim V = \dim W$: T is Iso $\Leftrightarrow T$ is Mono $\Leftrightarrow T$ is Epi.

4) $T: V \to W$ Iso $\Leftrightarrow \forall B$ basis of V, $T(B)$ basis of W.

Def :

let V. W be two F-vector spaces, we say that V and W are Iso.

If $\exists T: V \to W$, T is Iso.



Remark : "being Iso" is an equivalence relation.

$V \sim W \Leftrightarrow \exists T: V \to W$ Iso.

i) $V \sim V \Leftrightarrow Id: V \to V$ Iso ✓.

ii) If $V \sim W \Rightarrow \exists T: V \to W$. Iso $\Rightarrow \exists T^{-1}: W \to V$ Iso $\Rightarrow W \sim V$.

iii) If $V \sim W$ and $W \sim U \Rightarrow \exists T_1: V \to W$, $T_2: W \to U$ Iso $\Rightarrow \exists T_2 \circ T_1: V \to U$. Iso.

$\Rightarrow V \sim U$. (T_1, T_2 LM. $\Rightarrow T_2 \circ T_1$ LM. T_1, T_2 bij. $\Rightarrow T_2 \circ T_1$ bij.).

THEO :

let V, W two F-vector spaces $V \sim W \Leftrightarrow \dim V = \dim W$.

Proof :

$\Rightarrow$) We know that $\exists T: V \to W$ Iso. Then B basis of V. $\Rightarrow T(B)$ basis of W.

$\Rightarrow \dim V = \#B = \#T(B) = \dim W$.

$\Leftarrow$) Assume $\dim V = \dim W. = \#I$. take $B = \{v_i, i \in I\}$. basis of V. $B' = \{w_i, i \in I\}$ basis of W.

Define $T: V \to W$ st $T(v_i) = w_i$. $\Rightarrow T(\Sigma a_i v_i) = \Sigma a_i w_i$. $\Rightarrow T(B) = B'$. then T is Iso.

Example .

$V_1 = \mathbb{R}^2$, $V_2 = \mathbb{C}$. $\underline{V_3 = \{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \in M_2(\mathbb{R})\}}$ ← Iso as $\mathbb{R}$-vector spaces.

$T: V_1 \to V_2$ Iso, $(1,0) \to 1$. $(0,1) \to i$

$T(x,y) = T(x(1,0) + y(0,1)) = xT(1,0) + yT(0,1). = x + yi$.

$T': V_2 \to V_3$. Iso.

$1 \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. $i \to \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

$T'(a+bi) = aT'(1) + bT'(i). = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$

IMPORTANT REMARK :

$\dim V = \dim W \Rightarrow \exists T : V \longrightarrow W$ Iso.

This is not the same as saying that any LM is going to be Iso.

For example : $R^2 \sim R^2$, $Id : R^2 \rightarrow R^2$. Iso. $T(x,y) = (y,x)$ Iso.

BUT $T'(x,y) = (x,0)$ NOT Iso.

THEO:

let V, W be two F-vector space. $\dim V = n$. $\dim W = m$.

$Hom_F(V,W)$ is Iso to $M_{m\times n}(F)$.

Proof:

$\alpha : Hom_F(V,W) \rightarrow M_{m\times n}(F)$. $\alpha(T) = [T]_{B_1B_2}$, For B_1 basis of V. B_2 basis of W.

$\alpha(T_1+T_2) = [T_1+T_2]_{B_1B_2} = [T_1]_{B_1B_2} + [T]_{B_1B_2} = \alpha(T_1) + \alpha(T_2)$.

$\alpha(\lambda T) = [\lambda T]_{B_1B_2} = \lambda \cdot [T]_{B_1B_2} = \lambda \alpha(T)$. $\Rightarrow$ Check α is LM.

Mono = $\alpha(T) = 0 \Rightarrow T(v_j) = 0 \ \ \forall v_j \in B_1 \Rightarrow T = 0$.

Epi : $A = [a_{ij}]$: define $T(v_j) = \Sigma a_{ij} w_i \Rightarrow \alpha(T) = A$.

$\Rightarrow$ Therefore, α is Iso.

Corollary: $\dim Hom_F(V,W) = \dim V \cdot \dim W = n \cdot m$.

Example:

$T : F^n \longrightarrow F^m, \longleftrightarrow \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_n \end{cases} \Rightarrow T(x_1 \ldots x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n, \ldots, a_{m1}x_1 + \cdots + a_{mn}x_n)$.

$T(x) = (Ax^T)^T \qquad x^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad Ax^T = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$\Rightarrow [T]_{E_nE_m} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix} = A$. st

$T(1,0 \ldots 0) = (a_{11}, a_{21} \cdots a_{m1}) = a_{11}e_1 + \cdots + a_{m1}e_n$.

$T(0,1,\cdots 0) = (a_{12}, a_{22}, \cdots a_{m2}) = a_{12}e_1 + \cdots + a_{m2}e_n$.

$\cdots \ T(0 \cdots 0\ 1) = (a_{1n}, a_{2n}, \cdots a_{mn}) = a_{1n}e_1 + \cdots + a_{mn}e_n$

Connection: Systems $\leftrightarrow$ LM. LM $\longleftrightarrow$ Matrices. vectors $\longleftrightarrow$ matrices.

Def:

let V be an F-vector space, $B = \{v_1 \ldots v_n\}$. a basis of V.

We denote the matrix of V with respect to B as follows : $[v]_B \in M_{n\times 1}(F)$, $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ if $v = a_1v_1 + \cdots + a_nv_n$.

Example: $V = R^3$, $v = (2,1,0)$. $E_3 = \{e_1\ e_2\ e_3\}$. $B = \{(1,0,0), (0,1,0), (1,0,1)\}$

$[v]_{E_3} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad (2,1,0) = 2 \cdot (1,0,0) + 1 \cdot (0,1,0) + 0 \cdot (0,0,1)$.

$[v]_B = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \qquad (2,1,0) = 3(1,0,0) + 1(0,1,0) + (-1)(1,0,1)$

THEO:

let V be a F-vector space, $\dim V = n$. Then V is Iso to $M_{n\times 1}(F)$. And the map :

$V \xrightarrow{\beta} M_{n\times 1}(F)$. For B a basis of V. Is a Iso.

$v \mapsto [v]_B$

Proof:

let $B = \{v_1 \dots v_n\}$ $\beta(v) = [v]_B$. β is a LM

$\beta(v+w) = \begin{bmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \beta(v) + \beta(u)$.

$\beta(\lambda v) = \begin{bmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \lambda [v]_B = \lambda \beta(v)$.

$v = \sum a_i v_i$

$u = \sum b_i v_i$

$v+u = \sum (a_i+b_i) v_i$.

$\lambda v = \sum \lambda a_i v_i$.

$x = (x_1 \dots x_n)$.

$\downarrow$

$[x]_{\mathcal{E}_n} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{tikzcd} F^n \arrow[r, "T"] \arrow[d] & F^m \arrow[d] \\ M_{n\times1}(F) \arrow[r] & M_{m\times1}(F) \\ {[x]_{\mathcal{E}_n}} \arrow[r] & A[x]_{\mathcal{E}_n} = [y]_{\mathcal{E}_m} \end{tikzcd}$$

$\Rightarrow T(x_1 \dots x_n) = (a_{11}x_1 + a_{21}x_1 + \dots + a_{m1}x_1, \dots a_{1n}x_n + \dots + a_{mn}x_n) = (y_1, y_2 \dots y_m)$

$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \dots + a_{m1}x_1 \\ \vdots \\ a_{1n}x_n + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ $\Rightarrow A = [T]_{\mathcal{E}_n \mathcal{E}_m}$.

THEO: $\underset{M_{m\times1}(F)}{\underset{\cap}{[T(v)]_{B'}}} = \underset{M_{m\times n}(F)}{\underset{\cap}{[T]_{BB'}}} \cdot \underset{M_{n\times1}(F)}{\underset{\cap}{[v]_B}}$. $\Rightarrow {}^{m}[\quad] = {}^{m}\overset{n}{[\qquad]} \cdot [\quad]^{n}$

Proof:

$B = \{v_1 \dots v_n\}$. $B' = \{w_1 \dots w_m\}$. $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ $v = a_1v_1 + \dots + a_nv_n$.

$[T]_{BB'} = \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \vdots & & & & \\ b_{m1} & \cdots & b_{mj} & \cdots & b_{mn} \end{bmatrix}$. $T(v_j) = b_{1j}w_1 + \dots + b_{mj}w_m = \sum_{i=1}^{m} b_{ij} \cdot w_i$

$T(v) = T(a_1v_1 + \dots + a_nv_n) = a_1T(v_1) + \dots + a_nT(v_n) = a_1\sum b_{i1}w_i + a_2\sum b_{i2}w_i + \dots + a_n\sum b_{in}w_i$.

$= \sum_{j=1}^{n}\sum_{i=1}^{m} a_j b_{ij} w_i$. $= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} b_{ij}a_j\right) w_i \Rightarrow [T(v)]_{B'} = \begin{bmatrix} \sum_{j=1}^{n} b_{1j}a_j \\ \vdots \\ \sum_{j=1}^{n} b_{mj}a_j \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{m1} & \cdots & b_{mn} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ ☒

5.19 CLASS 20

Fix B, B' basis for V and W. $B = \{v_1 \dots v_n\}$. $B' = \{w_1 \dots w_m\}$

$\operatorname{Hom}_F(V,W) \xrightarrow[\alpha]{\text{Iso}} M_{m\times n}(F)$

$T \longmapsto [T]_{BB'} = \left[[T(v_1)]_{B'} \dots [T(v_n)]_{B'}\right] \Leftrightarrow T(v_j) = \sum a_{ij}w_i \Leftrightarrow [T(v_j)]_{B'} = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$.

$V \longrightarrow M_{n\times1}(F)$.

$v \longmapsto [v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \Longleftrightarrow v = a_1v_1 + \dots + a_nv_n$.

We have proven that α, β are LM.

$[T+T']_{BB'} = [T]_{BB'} + [T]_{BB'}$, $[\lambda T]_{BB'} = \lambda [T]_{BB'}$

$[v+v']_B = [v]_B + [v']_B$, $[\lambda v]_B = \lambda [v]_B$.

We also proved $V \xrightarrow{T_1} W \xrightarrow{T_2} U$.

$[T_2 \circ T_1]_{B_1B_3} = [T_2]_{B_2B_3} [T_1]_{B_1B_2}$

$$\begin{tikzcd} v \arrow[r] \arrow[ddd, bend right, "B"'] & T(v) \arrow[ddd, bend left, "B'"] \\ V \arrow[r, "T"] \arrow[d, leftrightarrow, "\beta_V"'] & W \arrow[d, leftrightarrow, "\beta_W"] \\ M_{n\times1}(F) & M_{m\times1}(F) \\ {[v]_B} \arrow[r] & {[T]_{BB'}[v]_B = [T(v)]_{BB'}} \end{tikzcd}$$

Example.

$T(x,y) = (x+y, x-y)$, $T: \mathbb{R}^2 \to \mathbb{R}^2 \Leftrightarrow \begin{cases} x+y = b_1 \\ x-y = b_2 \end{cases}$

$\mathcal{E}_2 = \{(1,0),(0,1)\}$. $B = \{(1,0),(1,1)\}$.

$[T]_{\mathcal{E}_2} = \left[[T(1,0)]_{\mathcal{E}_2} \ [T(0,1)]_{\mathcal{E}_2}\right] = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

Since $T(1,0) = (1,1) = 1(1,0) + 0(0,1)$ $\Rightarrow T(x,y) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$T(0,1) = (1,-1) = 1(1,0) + (-1)(0,-1).$

$v = (2,3) \in \mathbb{R}^2$. $[(2,3)]_{B_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ since $(2,3) = 2\cdot(1,0) + 3(0,1)$.

$T(v) = T(2,3) = (5,1)$ $\Rightarrow [T(2,3)]_{B_2} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ since $(5,1) = 5(1,0) + 1\cdot(0,1)$.

$\Rightarrow [T(2,3)]_{B_2} = [T]_{B_2} \cdot [(2,3)]_{B_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$[T]_{B_2 B} = \left[[T(1,0)]_B, [T(0,1)]_B \right] = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$ since $T(1,0) = (1,1) = 0(1,0) + 1(1,1)$. $T(0,1) = (1,-1) = 2(1,0) + (-1)(1,1)$.

$[v]_{B_2} = [(2,3)]_{B_2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$

$\Rightarrow [T(v)]_B = [T(2,3)]_B = [T]_{B_2 B} \cdot [(2,3)]_{B_2} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \Leftrightarrow T(2,3) = 6(1,0) + (-1)(1,1). = (5,1).$

Notation: $T : V \to V$.

$B \quad B' \longrightarrow [T]_{BB'}$

$B \quad B \longrightarrow [T]_B = [T]_{BB}$

$B' \quad B' \longrightarrow [T]_{B'B'} = [T]_{B'}$

THEO: $T \in \mathrm{Hom}_F(V,W)$. $\dim_F V = n$. $\dim_F W = n$. Then:

1) T Iso $\Leftrightarrow$ 2) $[T]_{BB'}$ is INVERTIBLE. $\forall BB'$ basis of V and W.

$\Leftrightarrow$ 3) $[T]_{BB'}$ INVERTIBLE for some B, B' basis of V and W.

Proof:

1) $\Rightarrow$ 2): We know that $T : V \to W$ is an Iso $\Rightarrow \exists T^{-1} : W \to V$, $T^{-1} \in \mathrm{Hom}_F(W,V)$. st $T \circ T^{-1} = Id_W$. $T^{-1} \circ T = Id_V$.

let B, B' basis of V, W. let's apply α.

$[T^{-1}]_{B'B} \cdot [T]_{BB'} = [T^{-1} \circ T]_{BB} = [Id_V]_{BB} = \left[[Id_V(v_1)]_B \cdots [Id_V(v_n)]_B \right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = Id_n$.

In the same way:

$[T]_{BB'} \cdot [T^{-1}]_{B'B} = [T \circ T^{-1}]_{B'B'} = [Id_W]_{B'} = Id_n$.

$\Rightarrow Id_n = [T]_{BB'} [T^{-1}]_{B'B} = [T^{-1}]_{B'B} \cdot [T]_{BB'}$ $\Rightarrow [T]_{BB'}$ is Invertible and $[T]^{-1}_{BB'} = [T^{-1}]_{B'B}$

Remark: $\overset{B}{V} \xrightarrow{T} \overset{B'}{W} \xrightarrow{T^{-1}} \overset{B}{V}$. $\Rightarrow [Id(v_1)]_B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\qquad \Rightarrow [Id(v_n)]_B = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

$v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$. $\qquad v_n = 0 \cdot v_1 + \cdots + 1 \cdot v_n$.

$\Rightarrow [Id(v_i)]_B = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \rightarrow i$

2) $\Rightarrow$ 3). ✓. True $\forall (BB')$ $\Rightarrow$ True for a particular pair (B, B').

3) $\Rightarrow$ 1): We know that $\exists B, B'$: $[T]_{BB'}$ Invertible. let $C = [T]_{BB'} \in M_n(F)$. $\Rightarrow \exists C^{-1} \in M_n(F)$: $C C^{-1} = Id_n = C^{-1} C$.

Using that α is an Iso, we know that $\exists S \in \mathrm{Hom}_F(W,V)$: $\alpha(S) = [S]_{B'B} = C^{-1}$.

$\alpha_{W,V} : \mathrm{Hom}_F(W,V) \longrightarrow M_{n \times n}(F)$.

$S \longmapsto [S]_{B'B} = C^{-1}$

$\alpha_{V,V} : \mathrm{Hom}_F(\overset{B}{V}, \overset{B}{V}) \to M_n(F)$

$\alpha_{W,W} : \mathrm{Hom}_F(\underset{B'}{W}, \underset{B'}{W}) \longrightarrow M_n(F)$

$\underset{B}{V} \xrightarrow{T} \underset{B'}{W} \xrightarrow{S} \underset{B}{V}.$

let's see that $S = T^{-1}$.

$[S\circ T]_{BB} = [S]_{B'B}[T]_{BB'} = C^{-1}\cdot C = Id_n = [Id_V]_{BB} \Rightarrow \alpha_{V,V}(S\circ T) = \alpha_{V,V}(Id_V) \overset{\alpha\ Iso}{\Longrightarrow} S\circ T = Id_V$.

$[T\circ S]_{B'B'} = [T]_{BB'}[S]_{B'B} = C\cdot C^{-1} = Id_n = [Id_W]_{B'B'} \Rightarrow \alpha_{WW}(T\circ S) = \alpha_{WW}(Id_W) \overset{\alpha\ Iso}{\Longrightarrow} T\circ S = Id_W$.

CHECK OF BASIS.

V, B_1, B_2 basis : $[v]_{B_1} \overset{?}{\longleftrightarrow} [v]_{B_2}$.

$V \xrightarrow{T} W$ $\quad [T]_{B_1B_1'} \overset{?}{\longleftrightarrow} [T]_{B_2B_2'}$. $\quad [B_1']_{B_2'} [T]_{B_1B_1'} \cdot [B_2]_{B_1} \cdot [v]_{B_1}$.

$B_1B_2 \quad B_1' B_2'$.

Def:

Let V be an F-vector space, B_1, B_2 two basis. The change of basis matrix from B_1 to B_2 is the matrix:

$$M(Id, B_1, B_2) = C(B_1, B_2) = [B_1]_{B_2} = [Id_V]_{B_1B_2} = \alpha(Id_V).$$

THEO: V, W F-vector spaces, B_1 B_2 basis of V, B_1' B_2' basis of W. $T \in Hom_F(V,W)$.

1) $[B_1]_{B_2}\cdot[B_2]_{B_1} = Id_n$. that is: $[B_1]_{B_2}^{-1} = [B_2]_{B_1}$.

2) $[B_1]_{B_2}\cdot[v]_{B_1} = [v]_{B_2}$.

3) $[B_1']_{B_2'}[T]_{B_1B_1'}[B_2]_{B_1} = [T]_{B_2B_2'}$. $\quad [C']_{B}[T]_{CC'} \quad = [T]_{CB}$.

Proof: $\quad [T]_{B'B'} = [B]_{B'}[T]_{BB}\cdot[B']_{B}$.

1) $[B_1]_{B_2}\cdot[B_2]_{B_1} = [Id_V]_{B_1B_2}\cdot[Id_V]_{B_2B_1} = [Id_V\circ Id_V]_{B_2B_2} = [Id_V]_{B_2} = Id_n$.

2) $[B_1]_{B_2}\cdot[v]_{B_1} = [Id_V]_{B_1B_2}\cdot[v]_{B_1} = [Id_V(v)]_{B_2} = Id_n$.

3) $[B_1']_{B_2'}\cdot[T]_{B_1B_1'}\cdot[B_2]_{B_1} = [Id_V]_{B_1'B_2'}\cdot[T]_{B_1B_1'}\cdot[Id_V]_{B_2B_1} = [Id_V\circ T]_{B_1B_2'}\cdot[Id_V]_{B_2B_1} = [T]_{B_1B_2'}\cdot[Id_V]_{B_2B_1}$
$= [T\circ Id_V]_{B_2B_2'} = [T]_{B_2B_2'}$.

Example 1: $T: R^2 \to R^3$, $[T]_{E_2E_3} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$

$[T(x,y)]_{E_3} = [T]_{E_2E_3}\cdot[(x,y)]_{E_2} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}$ (since $(x,y) = x(1,0) + y(0,1)$). $= \begin{bmatrix} x-y \\ 2y \\ x+3y \end{bmatrix}$

$T(x,y) = (x-y)(1,0,0) + 2y(0,1,0) + (x+3y)(0,0,1) = (x-y, 2y, x+3y)$.

Example 2:

Describe $T: R^2 \longrightarrow R^3$ st $[T]_{B_1B_2} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad B_1 = \{(1,0), (1,1)\}$. $B_2 = \{(1,0,0), (1,0,1), (0,1,1)\}$.

Option 1: $[T(x,y)]_{B_2} = [T]_{B_1B_2}\cdot[(x,y)]_{B_1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-y \\ y \end{bmatrix}$ since $(x,y) = (x-y)(1,0) + y(1,1)$

$= \begin{bmatrix} x-2y \\ 2x-y \\ x-y \end{bmatrix}$

$T(x,y) = (x-2y)(1,0,0) + (2x-y)(1,0,1) + (x-y)(0,1,1). = (x-3y, x-y, 3x-2y)$.

Option 2:

$[T]_{E_2E_3} = [B_2]_{E_3}[T]_{B_1B_2}[E_2]_{B_1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}$: $[T(x,y)]_{E_3} = [T]_{E_2E_3}[(x,y)]_{E_2} = \begin{bmatrix} 3 & -3 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow T(x,y) = (3x-3y, x-y, 3x-2y)$.

PROJECTIONS.

Def: A projection is a LM $P: V \to V$ st $P\circ P = P$.

LEMMA: P is a projection $\Leftrightarrow P(w) = w \quad \forall w \in ImP$.

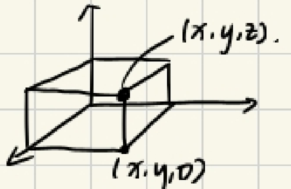
Proof: $\Rightarrow)$ $w \in ImP \Rightarrow w = P(v) \Rightarrow P(w) = P(P(v)) = P(v) = w$.

$\Leftarrow)$ $P\circ P(v) = P(\underbrace{P(v)}_{w \in ImP}) = P(v)$

THEO: $P: V \to V$ LM.

1) $P \circ P = P \Rightarrow V = \ker p \oplus \operatorname{Im} P$.

2) If $V = S \oplus T$, then $\exists! \, p: V \to V$ st $\ker p = S$ and $\operatorname{Im} p = T$.



$p(x,y,z) = (x,y,0)$.

$P \circ P(x,y,z) = P(x,y,0) = (x,y,0) = P(x,y,z)$.

5.21. CLASS 21.

Def: $p: V \to V$ is a Projection if P is a LM st $P \circ P = P$.

Proof: P is projection $\Leftrightarrow p(w) = w \;\; \forall w \in \operatorname{Im} p$.

THEO:

1) P is Projection then $V = \ker p \oplus \operatorname{Im} p$.

2) if $V = S \oplus U$ then $\exists! \, p: V \to V$ a projection st $S = \ker p$ and $U = \operatorname{Im} p$.

Proof:

1) $\ker p \subseteq V$. $\operatorname{Im} p \subseteq V$. subspaces $\Rightarrow \ker p + \operatorname{Im} p \subseteq V$.

let $v \in V$. $\Rightarrow v = p(v) + v - p(v)$ $\quad p(v) \in \operatorname{Im} p$. $v - p(v) \in \ker p$. And $p(v - p(v)) = p(v) - p\circ p(v) = p(v) - p(v) = 0$

$\Rightarrow V \subseteq \ker p + \operatorname{Im} p$. $\Rightarrow V = \ker p + \operatorname{Im} p$.

let $w \in \ker p \cap \operatorname{Im} p$. $\Rightarrow p(w) = 0$ and $w = p(v)$ $v \in V$ Then $0 = p(w) = p(p(v)) = p(v) = w$ $\Rightarrow w = 0$. $\Rightarrow \ker p \cap \operatorname{Im} p = \{0\}$

$\Rightarrow V = \ker p \oplus \operatorname{Im} p$. ∎

2) We have to define $p: V \to V$ st $\begin{cases} P \text{ is LM.} \\ P \text{ is a projection.} \\ \ker p = S. \\ \operatorname{Im} p = U. \end{cases}$ $\qquad P(v) = P(s+u) = P(s) + p(u) = 0 + u \Rightarrow P$ is Unique.

P is LM:

$P(v_1 + v_2) = p(s_1 + u_1 + s_2 + u_2) = p(\overset{\in S}{s_1 + s_2} + \overset{\in U}{u_1 + u_2}) = u_1 + u_2 = P(v_1) + p(v_2)$.

$p(\lambda v) = p(\lambda(s+u)) = p(\overset{\in S}{\lambda s} + \overset{\in U}{\lambda u}) = \lambda u = \lambda \cdot p(v)$.

P is Projection:

$(P \circ P)(v) = (p \circ p)(s+u) = p(p(s+v)) = p(u) = p(0+u) = p(v)$, $\forall v \Rightarrow P \circ P = P$.

$\ker p = \{v : p(v) = 0\} = \{v = s+u : p(v) = p(s+u) = u = 0\} = \{v = s+u : u = 0\} = S$.

$\operatorname{Im} p = \{p(v), v \in V\} = \{p(v) = p(s+u) = u, v \in V\} = U$.

Corollary: let $p: V \to V$ be projection. Then $\exists B$ a basis of V. st:

$$[p]_B = \begin{bmatrix} 1 & \cdots & & 0 \\ & \ddots\, 1 & 0 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & 0 \end{bmatrix} = \begin{bmatrix} Id_r & 0 \\ 0 & 0 \end{bmatrix} \in M_n(F) \quad \text{if } \dim V = n.$$

Proof:

We know $V = \ker p \oplus \operatorname{Im} p$. Take $\{v_1 \cdots v_r\}$ a basis of $\operatorname{Im} p$. And $\{v_{r+1} \cdots v_n\}$ a basis of $\ker p$.

$\Rightarrow B = \{v_1 \cdots v_r, v_{r+1} \cdots v_n\}$ is a basis of V.

$$[p]_B = \Big[[p(v_1)]_B \cdots [p(v_r)]_B, \underset{\overset{\shortparallel}{0}}{[p(v_{r+1})]_B} \cdots \underset{\overset{\shortparallel}{0}}{[p(v_n)]_B} \Big] = \Big[[v_1]_B \, [v_2]_B \cdots [v_r]_B \Big] = \begin{bmatrix} \begin{bmatrix} 1 & \cdots & 0 \\ 0 & 1 \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix} & \begin{matrix} 0 & \cdots & 0 \end{matrix} \\ \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} 0 & \cdots & 0 \end{matrix} \end{bmatrix}$$

Remark:

1) THE PREVIOUS COROllARY CAN BE GENENAlIZED AS FOllOWING :

P is a projection $\Leftrightarrow \exists B$ is a basis st $[P]_B = \begin{bmatrix} Id & \cdots 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow)$ ✓.

$\Leftarrow)$ We need to prove $P \circ P = P$. $B = \{v_1 \dots v_r, v_{r+1} \dots v_n\}$. $[P]_B = \left[[P(v_1)]_B \cdots [P(v_r)]_B, \overbrace{[P(v_{r+1})]_B \cdots [P(v_n)]_B}^{0} \right]$

$\Rightarrow P(v_i) = v_i \ \forall i \in \{1 \dots r\}$. $P(v_j) = 0 = v_j = 0 \ \forall j \in \{r+1 \dots n\}$. $\Rightarrow P(v) = v \ \forall v \in V$. $\Rightarrow P \circ P(v) = P(v) \ V$.

2) THE PROJECTIONS COROllARY CAN BE GENERALIZED TO ANY LM $T: V \to W$. BUT NEED TWO BASIS.

THEO:

Let $T: V \to W$. a LM, $\dim \operatorname{Im} T = r$. Then $\exists B$ a basis for V and B' a basis for W st.

$$[T]_{BB'} = \begin{bmatrix} \overset{r}{{}_r Id_r} & \overset{n-r}{0_{r \times n-r}} \\ 0_{m-r \times r} & 0_{n-r \times m-r} \end{bmatrix} \in M_{m \times n}(F). \quad \text{if } \dim V = n. \ \dim W = m.$$

Proof: Since $\dim \operatorname{Im} T = r$ we know $\dim \ker T = \dim V - \dim \operatorname{Im} T = n-r$. Take $\{v_{r+1}, \dots v_n\}$ a basis of $\ker T$.

$\{v_{r+1} \dots v_n\}$ LI in V. we can extend it to a basis of B of V: $B = \{v_1 \dots v_r, v_{r+1} \dots v_n\}$.

We know $\operatorname{Im} T = \{T(v), v \in V\} = \{T(\sum_{i=1}^{n} a_i v_i) = a_1 T(v_1) + a_2 T(v_2) + \dots + a_r T(v_r) + a_{r+1} T(v_{r+1}) + \dots + a_n T(v_n), a_i \in F\} = \{a_1 T(v_1) + \dots + a_r T(v_r)\}$.

$= \operatorname{span}(T(v_1), T(v_2), \dots T(v_r))$. and $\dim \operatorname{Im} T = r \Rightarrow \{w_1 = T(v_1), w_2 = T(v_2), \dots w_r = T(v_r)\}$ is a basis for $\operatorname{Im} T$.

$\Rightarrow \{w_1 \dots w_r\}$ is LI in W. we can extend it to a basis $B' = \{w_1 \dots w_r, w_{r+1} \dots w_m\}$ a basis for W.

$$[T]_{BB'} = \left[[T(v_1)]_{B'} \ [T(v_2)]_{B'} \cdots [T(v_r)]_{B'} \ [T(v_{r+1})]_{B'} \cdots [T(v_n)]_{B'} \right] = \left[[w_1]_{B'} \ [w_2]_{B'} \cdots [w_r]_{B'} \ [0]_{B'} \cdots [0]_{B'} \right].$$

$$= \begin{bmatrix} \begin{matrix} 1 & 0 \cdots 0 \\ \vdots & \ddots \\ 0 & 1 \end{matrix} & 0 \\ 0 & 0 \end{bmatrix} \quad \blacksquare$$

DETERMINANT OF MATRICES.

$\det: M_n(F) \to F$. function with properties.

$\swarrow$ option 1.

Define the function

Explicitely then prove properties.

$\gcd(a,b) = MAX(Div(a) \cap Div(b))$.

$\searrow$ option 2. (Categorical option, universal properties).

Define a function by its properties. Then prove that this function exists and is unique.

$d = \gcd(a,b)$ if — $d|a \wedge d|b$

if $d'|a \wedge d'|b \Rightarrow d'|d$.

OPTION 1:

$$\det A = |A| = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \Sigma(\pm 1) \text{ products of } n \text{ elements belonging to different rows and columns.}$$

$$= \sum_{\sigma \in S_n} \operatorname{sign} \sigma \ a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n-1\sigma(n-1)} a_{n\sigma(n)}. \qquad n = 1, 2, 3.$$

$\sigma(1) = 1, \dots n$.

$\sigma(2) \neq \sigma(1)$.

$\sigma(3) \neq \sigma(2), \sigma(1)$.

$\vdots$

$\sigma(n-1) \neq \sigma(1), \dots \sigma(n-2)$.

$\sigma = \{1 \dots n\} \longrightarrow \{1, \dots n\}$. bijection.

$S_n = \{\sigma: \{1 \dots n\} \to \{1 \dots n\} \text{ bijection}\}$

$n=1$. $\det : M_1(F) \to F$ $\det[a] = a$.

$n=2$. $\det : M_2(F) \to F$.

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \underset{\substack{\downarrow \\ \sigma(1)=1 \ \sigma(2)=2 \\ \sigma:\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}}}{a_{11}a_{22}} - \underset{\substack{\downarrow \\ \sigma(1)=2 \ \sigma(2)=1. \\ \sigma:\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.}}{a_{12}a_{21}}.$$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

$n=3$. $\det : M_3(F) \to F$. st:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underset{\sigma\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}}{a_{11}\cdot a_{22}\cdot a_{33}} - \underset{\sigma\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.}{a_{12}a_{21}a_{33}}. + \underset{\sigma:\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}{a_{12}a_{23}a_{31}} - \underset{\sigma\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}{a_{13}a_{22}a_{31}} + \underset{\sigma\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}{a_{13}a_{21}a_{32}} - \underset{\sigma\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.}{a_{11}a_{23}a_{32}}.$$

OPTION 2.

Def: a function $D: M_n(F) \to F$. is called multilinear if $D(A) = D(R_1, R_2 \ldots R_n)$.

$D(R_1R_2 \ldots R_{i-1}, R_i' + R_i'', R_{i+1}, \ldots R_n) = D(R_1R_2 \cdots R_i' R_{i+1} \cdots R_n) + D(R_1 \cdots R_{i-1} R_i'', R_{i+1} \cdots R_n)$.

$D(R_1 \ldots R_{i-1}, \lambda R_i, R_{i+1} \cdots R_n) = \lambda D(R_1 \cdots R_i, R_{i+1} \cdots R_n)$. $\forall i$.

Def: $D: M_n(F) \to F$ is ALTERNATING if:

1) $D(R_1 \cdots R_i \cdots R_j \ldots R_n) = -D(R_1 \cdots R_j \cdots R_i \ldots R_n)$. $i \neq j$.

2) $D(R_1 \ldots R_i \ldots R_i \ldots R_n) = 0$.

Def: $\det: M_n(F) \to F$ is a MULTILINEAR ALITERNATING. function st $\det(Id) = 1$.

$n=2$. $\det(A) = D(R_1, R_2) = D(a_{11}e_1 + a_{12}e_2,\ a_{21}e_1 + a_{22}e_2) = a_{11}D(e_1, a_{21}e_1 + a_{22}e_2) + a_{12}D(e_2, a_{21}e_1 + a_{22}e_2)$.

$$= a_{11}a_{21}\underset{\substack{\| \\ D(Id)=1}}{D(e_1, e_2)} + a_{11}a_{22}\underset{\substack{\| \\ D(Id)=1}}{D(e_1, e_2)} + a_{12}a_{21}\underset{\substack{\| \\ -D(Id)=-1}}{D(e_2\, e_1)} + a_{12}a_{22}\underset{\substack{\| \\ 0.}}{D(e_2, e_2)}.$$

$= a_{11}a_{22} - a_{12}a_{21}$.

Now we have to prove that a function satisfying these 3 properties. Exists and Unique.

THEO: If a function det exists, it is unique.

Proof:

$\det A = \det(R_1 \ldots R_n) = \det(\sum a_{1i}e_i, \sum a_{2i}e_i, \ldots \sum a_{ni}e_i)$

$= \sum a_{1i}\det(e_i, \sum a_{2i}e_i, \ldots \sum a_{ni}e_i) = \sum^{i_1}\sum^{i_2} a_{1i_1}a_{2i_2}\det(e_{i_1}, e_{i_2}, \sum a_{3i}e_i \ldots \sum a_{ni}e_i)$.

$= \sum\sum \cdots \sum^{i_n} a_{1i_1}a_{2i_2} \cdots a_{ni_n} D(e_{i_1}, e_{i_2}, \cdots e_{i_n}) = \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)} D(e_{\sigma(1)}, \cdots e_{\sigma(n)})$.

$= \sum_{\sigma \in S_n} \operatorname{sign}\sigma\ a_{1\sigma(1)} \cdots a_{n\sigma(n)} \overset{\substack{1 \\ \|}}{D(e_1, e_2 \cdots e_n)} = \sum_{\sigma \in S_n} \operatorname{sign}\sigma \cdot a_{1\sigma(1)} \cdots a_{n\sigma(n)}$. So it is unique.

THEO:

The function $\det: M_n(F) \to F$ satisfying MOLTILINEAR, ALTERNATING and $\det Id = 1$. Exists.

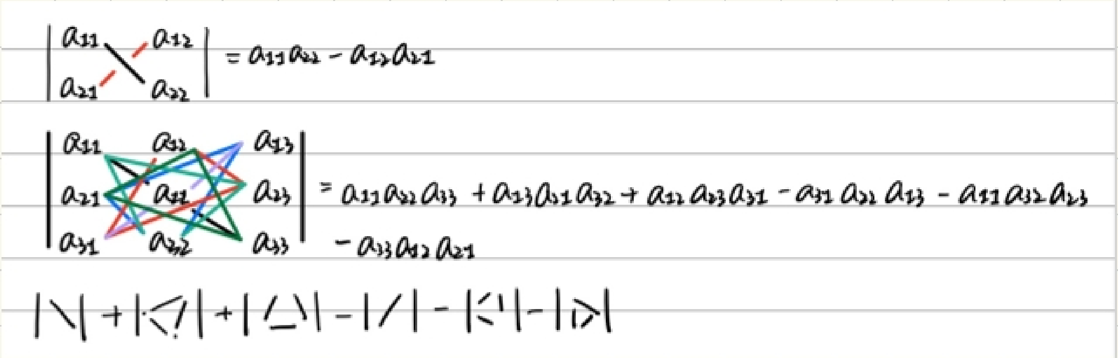
Proof: By induction.

$n=1$ $\det: M_1(F) \to F$. $\det[a] = a$. is a determinant function: Linear Alternating. $\det Id = 1$.

Assume that $\det: M_{n-1}(F) \to F$ exists, we can check that:

$\det_n A := \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det_{n-1} A^{(i,j)}$ for some fixed i. where $A^{(i,j)} = A$ without the row and the column where a_{ij} belongs.

$$A^{(i,j)} = \left(\overset{|}{\underset{|}{-a_{ij}-}} \right) \in M_{n-1}(F).$$



$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{33}a_{12}a_{21}$$

5.26. ClASS 22

Def: det: $M_n(F) \to F$ is a Multilinear, Althrnating function st $\det(Id) = \det(e_1, e_2, \cdots e_n)$.

Multilinear:

- $\det(R_1 \ldots R_i + R_i', \ldots R_n) = \det(R_1, \ldots R_i \ldots R_n) + \det(R_1 \cdots R_i', \cdots R_n)$
- $\det(R_1 \ldots aR_i \ldots R_n) = a \cdot \det(R_1 \cdots R_i \cdots R_n)$.

Alternating:

- $\det(R_1 \ldots R_i \ldots R_j \ldots R_n) = -\det(R_1 \cdots R_j \cdots R_i \ldots R_n)$. $\quad i \neq j$.
- $\det(R_1 \ldots R_i \ldots R_i \ldots R_n) = 0$

Example: $\det A = |A|$.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+a_{21}' & a_{22}+a_{22}' & a_{23}+a_{23}' \\ b\cdot a_{31} & b\cdot a_{32} & ba_{33} \end{vmatrix} = b\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + b\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Problem: Does det EXIST IS IT UNIQUE?

THEO 1: if a function det exist, then is unique.

Proof:

if $D: M_n(F) \to F$ Multilinear, Alternating, $D(Id) = 1$. then:

$$D\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{pmatrix} = D((a_{11} \cdots a_{1n}), (a_{21} \cdots a_{2n}) \ldots (a_{n1} \cdots a_{nn})). = D\left(\sum_{i_1=1}^n a_{1i_1}e_{i_1}, \sum_{i_2=1}^n a_{2i_2}e_{i_2}, \ldots \sum_{i_n=1}^n a_{ni_n}e_{i_n}\right).$$

$$= \sum_{i_1=1}^n a_{1i_1} D\left(e_{i_1}, \sum a_{2i_2}e_{i_2}, \ldots \sum a_{ni_n}e_{i_n}\right) = \sum_{i_1=1}^n a_{1i_1} \sum_{i_2=1}^n a_{2i_2} D\left(e_{i_1}, e_{i_2} \ldots \sum a_{ni_n}e_{i_n}\right).$$

$$= \sum\sum\cdots\sum_{i_n=1}^n a_{1i_1}a_{2i_2}\cdots a_{ni_n} \underline{D(e_{i_1} \ldots e_{i_n})} \xrightarrow{} \neq 0. \quad \{\overset{\sigma(1)}{\overset{\downarrow}{i_1}} \cdots \overset{\sigma(n)}{\overset{\downarrow}{i_n}}\} = \{1, 2, \ldots n\} \Rightarrow = \sum_{\sigma \in S_n} a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)} D(e_{\sigma(1)}, \cdots e_{\sigma(n)}).$$

$$= \sum_{\sigma \in S_n} (\text{sign}) \cdot a_{1\sigma(1)} \cdot a_{2\sigma(2)} \cdots a_{n\sigma(n)} \quad \Rightarrow D \text{ is unique}.$$

THEO 2: The function det exists. Moreover, $\det A = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det A^{(i,j)} = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A^{(i,j)}$

Proof:

$\det_1: M_1(F) \to F$, $\det_1([a]) = a$. is a determinant. By induction, assume $\det_{n-1}$ exists

define: $\Psi_1: M_n(F) \to F$. $\Psi_2: M_n(F) \to F$. by the formulas $\Psi_1(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det_{n-1} A^{(i,j)}$ $\quad \Psi_2(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det_{n-1} A^{(i,j)}$.

check that Ψ_1, Ψ_2 are Multi Alter, $\Psi_1(Id) = 1$. $\Psi_2(Id) = 1$ $\Rightarrow \Psi_1$ and Ψ_2 are determinant function and by theo 1. $\Psi_1(A) = \Psi_2(A)$ since Det of A is unique. ∎

$$A^{(i,j)} = \begin{pmatrix} a_{11} & \cdots & \vert & \cdots & a_{1n} \\ \text{—} & & a_{ij} & & \text{—} \\ a_{n1} & \cdots & \vert & \cdots & a_{nn} \end{pmatrix} \Rightarrow \text{remove } R_i \text{ and } C_j.$$

Example:

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

φ_1: $j=3$.

φ_2: $i=2$.

φ_1: $(-1)^{1+3}a_{13}\det A^{(1,3)} + (-1)^{2+3}a_{23}\det A^{(2,3)} + (-1)^{3+3}a_{33}\det A^{(3,3)} + (-1)^{4+3}a_{43}\det A^{(4,3)}$

$$= 0.\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} - 0.\begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + (-1).\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} - 1.\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} - \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

use φ_1.

use φ_2.

$$= (-1).\left[(-1)^{3+1}.0.\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + (-1)^{3+2}.1.\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + (-1)^{3+3}.0.\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}\right] + (-1).\left[(-1)^{1+2}.0.\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + (-1)^{2+2}.1.\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + (-1)^{3+2}.0\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\right]$$

$$= \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 2\cdot1 - 1\cdot0 - (2\cdot0 - 1\cdot1) = 2+1 = 3.$$

HOW DOES det work with row operation?

THEO: $A \in M_n(F)$.

1) $A \xrightarrow{R_i \longleftrightarrow R_j} A'$ then $\det A = -\det A'$

2) $A \xrightarrow{R_i \longleftrightarrow aR_i} A'$ $\forall a \neq 0$. then $\det A = a\cdot\det A'$.

3) $A \xrightarrow{R_i \longleftrightarrow R_i + aR_j} A'$ $i \neq j$. then $\det A = \det A'$.

Proof:

$$A = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix} \quad A' = \begin{pmatrix} R_1' \\ \vdots \\ R_n' \end{pmatrix}$$

1) $\det A' = \det(R_1 \ldots R_j \ldots R_i \ldots R_n) = -\det(R_1 \ldots R_i \ldots R_j \ldots R_n) = -\det A$.

2) $\det A' = \det(R_1 \ldots aR_i \ldots R_n) = a\cdot\det(R_1 \ldots R_i \ldots R_n). = a\det A$.

3) $\det A' = \det(R_1 \ldots R_i + aR_j \ldots R_j \ldots R_n) = \det(R_1 \cdots R_i \ldots R_j \ldots R_n) + \det(R_1 \cdots aR_j \ldots R_j \ldots R_n) = \det A + 0 = \det A$.

Example:

$$\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix} \overset{R_1 \leftrightarrow R_3}{=} -\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} \overset{R_3 = R_3 - 2R_1}{=} -\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix} \overset{R_4 = R_4 - R_2}{=} -\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{vmatrix} \overset{R_3 \leftrightarrow R_4}{=} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{vmatrix} \overset{R_4 \to R_4 - 2R_3}{=} \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 3\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3|Id| = 3.$$

THEO:

1) $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ & & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$

2) $\det A = \det A^T$

3) $\det(A\cdot B) = \det A \cdot \det B$. (Find A, B $\det(A+B) \neq \det A + \det B$).

4) A Invertible $\Leftrightarrow$ $\det A \neq 0$.

In this case, $A^{-1} = \frac{1}{\det A}\cdot \mathrm{Adj}A$, $\mathrm{Adj}A$ = Adjoint of A.

Proof:

1) by induction: $|a_{11}| = a_{11}$ then for n-1. $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \\ 0 & & \cdots & a_{nn} \end{vmatrix} = (-1)^{1+1}\cdot a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \\ 0 & & a_{nn} \end{vmatrix} = (-1)^{1+1}\cdot a_{11}\cdot(-1)^{1+1}\cdot a_{22} \begin{vmatrix} a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \\ 0 & & a_{nn} \end{vmatrix}$

$\Rightarrow$ induction $= a_{11}\cdot a_{22}\cdots a_{nn}$.

2). by induction: $[a_n]^T = [a_{11}]$ ✓. Take for n-1: $\det A = \varphi_1(A) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det A^{(i,j)} = \sum_{i=1}^{n}(-1)^{j+i}(A^T)_{ji}\det(A^{(i,j)})^T$.
$= \sum_{j=1}^{n}(-1)^{j+i}(A^T)_{ji}\det(A^T)^{(j,i)} = \varphi_2(A^T) = \det A^T$.

3). let $D: M_n(F) \to F$ by given by $D(R_1 \ldots R_n) = \det(R_1B, R_2B, \cdots R_nB)$.

a) D is Multi:

$D(R_1 \ldots R_i + R_i' \ldots R_n) = \det(R_1B \ldots (R_i+R_i')B, \ldots R_nB) = \det(R_1B \ldots R_iB \ldots R_nB) + \det(R_1B \cdots R_i'B \cdots R_nB)$.

$D(R_1 \cdots aR_i \cdots R_n) = \det(R_1B \cdots aR_iB \ldots R_nB) = a\cdot\det(R_1B \cdots R_iB \cdots R_nB) = a\cdot D(R_1 \ldots R_n)$.

b) D is Alter:

$D(R_1 \ldots R_j \ldots R_i \ldots R_n) = \det(R_1B \ldots R_jB \ldots R_iB \ldots R_nB) = -\det(R_1B, \ldots R_iB \cdots R_jB \ldots R_nB) = -\det(R_1 \ldots R_i \ldots R_j \ldots R_n)$.

$D(R_1 \ldots R_i \ldots R_i \ldots R_n) = \det(R_1B \ldots R_iB \ldots R_iB \ldots R_nB) = 0$.

4).

$\Rightarrow$) if A is invertible $\Rightarrow \exists A^{-1}$ st $A\cdot A^{-1} = Id \Rightarrow 1 = \det(Id) = \det(A\cdot A^{-1}) \overset{(3)}{=} \det(A)\cdot\det(A^{-1}) \Rightarrow \det A \neq 0$.

$\Leftarrow$) $\mathrm{Adj}A = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$ $b_{ij} = (-1)^{i+j}\cdot\det_{n-1}A^{(j,i)}$ we will prove that $A\cdot\mathrm{Adj}A = \det A\cdot Id$.

$\Rightarrow A\cdot 1/\det A\cdot\mathrm{Adj}A = Id$.

$$\underset{A}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}} \cdot \underset{\mathrm{Adj}(A)}{\begin{pmatrix} (-1)^{1+1}|A^{(1,1)}| & (-1)^{2+1}|A^{(2,1)}| & \cdots \\ \vdots & \vdots & \\ (-1)^{1+n}|A^{(1,n)}| & (-1)^{2+n}|A^{(2,n)}| & \end{pmatrix}} = \begin{pmatrix} \sum_{i=1}^{n}(-1)^{1+i}a_{1i}|A^{(1,i)}| = \varphi_2(A) = \det A. & 0 & \cdots & 0 \\ \underline{\sum_{i=1}^{n}(-1)^{1+i}a_{2i}|A^{(1,i)}| = 0.} & \det A & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{n}(-1)^{1+i}a_{ni}|A^{(1,i)}| = 0 & \cdots & 0 \cdots & \det A \end{pmatrix}$$

$$A\cdot\mathrm{Adj}(A) = \det A\cdot I$$

Since $0 = \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} \underset{\varphi_2}{=} \sum_{j=1}^{n}(-1)^{1+j}a_{2j}|A^{(1,j)}| \longrightarrow A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$.

$0 = \begin{vmatrix} a_{n1} & \cdots & a_{nn} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{nn} & \cdots & a_{nn} \end{vmatrix} \underset{\varphi_2}{=} \sum_{j=1}^{n}(-1)^{n+j}a_{nj}|A^{(n,j)}|$

$\sigma: \{1, \ldots n\} \to \{1 \ldots n\}$.

$\sigma(1) = i_1$, $\sigma(2) = i_2$.

$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix}$.

5.28. CLASS 23.

EIGENVALUES AND EIGEN VECTORS.

$T: V \to V$ operators or endomordhisms.

$\downarrow$ 2M.

Question: Can we find a basis B of V st $[T]_B = [T]_{BB} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ diagonal matrix? But Not always.

if yes, many computations can be easily done.

$B = \{v_1 \dots v_n\} \Rightarrow = \left[(T(v_1))_B \dots (T(v_n))_B \right]$ $T(v_1) = \lambda_1 v_1$ $T(v_2) = \lambda_2 v_2$ $\dots$ $T(v_n) = \lambda_n v_n$.

$\Downarrow$ then we have.

$\ker T = \langle v \in V : T(v) = 0 \rangle = \{ \sum_{i=1}^{n} a_i v_i : T(\sum a_i v_i) = \sum a_i T(v_i) = \sum a_i \lambda_i v_i = 0 \} = \{ \sum a_i v_i : a_i \lambda_i = 0 \ \forall i \}$.

$= \langle \{ v_i : \lambda_i = 0 \} \rangle$.

$\operatorname{Im} T = \langle T(v_1), T(v_2) \dots T(v_n) \rangle = \langle \lambda_1 v_1 \dots \lambda_n v_n \rangle = \langle \{ \lambda_i v_i : \lambda_i \neq 0 \} \rangle. = \langle \{ v_i : \lambda_i \neq 0 \} \rangle$

T is Iso $\Leftrightarrow [T]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ invertible $\Leftrightarrow 0 \neq \det \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \lambda_1 \lambda_2 \dots \lambda_n \Leftrightarrow \lambda_i \neq 0. \ \forall i.$

In this case, $[T^{-1}]_B = [T]_B^{-1} = \begin{bmatrix} \lambda_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{-1} \end{bmatrix}$

Problem:

1) Given T, then do we know if the answer Yes or No?
How do we know if we can find B: $[T]_B$ is diagonal?
2) if the answer is Yes. how can find B?
3) if is No, which is the simplest matrix can associate to T?

$\downarrow$ Jordan form: can always find a basis B:

$$[T]_B = \begin{bmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & \cdots & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{matrix}} & & & 0 \\ & \boxed{\begin{matrix} \lambda_2 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & & \lambda_2 \end{matrix}} & & \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} \lambda_n & 1 & \cdots & 0 \\ & \ddots & \ddots & 1 \\ 0 & & & \lambda_n \end{matrix}} \end{bmatrix}$$

$\leftarrow$ see it in Algebra B.

$\exists B : [T]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \Leftrightarrow \exists B : T(v_i) = \lambda_i v_i \ \forall v_i \in B \Leftrightarrow \exists B : T(a v_i) = a T(v_i) = a \lambda_i v_i \ \forall v_i \in B.$

$\Leftrightarrow T(\langle v_i \rangle) \subset \langle v_i \rangle \ \forall v_i \in B.$

Def: Given $T \in \operatorname{Hom}_F(V, V)$. A subspace S of V is called INVARIANT UNDER T, or T-invariant. if $T(S) \subseteq S$.

Example.

1) $T: V \to V$, $T(0) = 0$. $\Rightarrow T(\langle 0 \rangle) = T(\{0\}) = \{0\} \subseteq \langle 0 \rangle$. $\Rightarrow \langle 0 \rangle$ is T-invarient $T(v) \in V. \ \forall v \in V.$
$\Rightarrow T(v) \subseteq V$ V is T-invariant.

2) $O: V \to V$. $O(v) = 0 \ \forall v \in V$. $\Rightarrow O(S) = \{ O(v), v \in S \} = \{0\} \subseteq S$. $\Rightarrow S$ is O-invarient. $\forall S \subset V$.
$Id: V \to V$. $Id(v) = v \ \forall v \in V$. $\Rightarrow Id(S) = \{ Id(v), v \in S \} = S$. $\Rightarrow S$ is Id-invarient $\forall S \subset V$.

3) $T: R[x] \to R[x]$. $T(p(x)) = P'(x)$.
then $T(a_0 + a_1 x + \dots + a_n x^n) = a_1 + 2a_2 x + 3a_3 x^2 + \dots + n a_n x^{n-1}$. $\Rightarrow T(R_{\leq n}[x]) \to T(R_{\leq n-1}[x]) \subseteq R_{\leq n}[x] \to R_{\leq n}[x]$ is T-invarient.

4) $T(x, y) = (x+y, x+y)$. $T(x, x) = (2x, 2x) = 2(x, x)$.
let $S = \{ (x, x), x \in F \}$. $T(S) \subseteq S$. $\Rightarrow S$ is T-invarient.

$U=\{(x,2x), x\in F\}$, $T(x,2x)=(3x,3x)$. $\Rightarrow T(1,2)=(3,3)\notin U$. $T(U)\nsubseteq U$. U is not T-invarient.

Finding B st $[T]_B$ is diagonal is equal to finding B: $\langle v_i\rangle$ is T-invarient $\forall v_i\in B$.

Def: Given $T\in Hom_F(V,V)$. A scalar $\lambda\in F$ is called an EIGENVALUE of T if $\exists v\in V$. $\underline{v\neq 0}$. st $T(v)=\lambda\cdot v$. In this case, the NON-ZERO VECTOR v is called an EIGENVECTOR associated to the EIGENVALUE λ.

if $v=0$. $\forall\lambda$ we have: $T(v)=T(0)=0=\lambda.0=\lambda v$

$E(\lambda,T)=\underset{\uparrow}{V_\lambda}=\{v\in V: T(v)=\lambda\cdot v\}=\{$Eigenvectors associated to $\lambda\}\cup\{0\}$
(charecteristic space associated to λ).

Example:

$T:F^2\to F^2$. $T(x,y)=(x+y,x+y)$.

EIGENVALUES: $\lambda\in F: T(x,y)=\lambda\cdot(x,y)$. for some $(x,y)\neq(0,0)$.

$$T(x,y)=\lambda(x,y)\Leftrightarrow(x+y,x+y)=(\lambda x,\lambda y).\Leftrightarrow\begin{cases}x+y=\lambda x\\ x+y=\lambda y\end{cases}\Rightarrow\lambda x=\lambda y\Rightarrow\lambda(x-y)=0\begin{cases}\lambda=0\to x=-y.\\ x=y\Rightarrow 2x=\lambda x\Rightarrow x=0\text{ or }\lambda=2\end{cases}$$

- $\lambda=0$. $T(x,x)=0\cdot(x,-x)$

  $\Rightarrow\lambda=0$ is an Eigenvalue, and $(x,-x)\neq0$ is an Eigenvector associated to $\lambda=0$.

  $V_0=\{(x,-x)\ \forall x\in F\}$

- $\lambda=2$: $T(x,x)=(2x,2x)=2(x,x)$.

  $\Rightarrow\lambda=2$ is an Eigenvalue, and $(x,x)\neq0$ is an Eigenvector associated to $\lambda=2$.

  $V_2=\{(x,x): x\in F\}$.

Take $(1,-1)\in V_0$. $(1,1)\in V_2$. $\Rightarrow\{(1,1),(1,-1)\}$ is a basis of F^2. $[T]_B=\begin{bmatrix}0&0\\0&2\end{bmatrix}$.

since $T(1,-1)=0$. $T(1,1)=2(1,1)=0\cdot(1,-1)+2(1,1)$. ✓.

Proposition: if λ is an Eigenvalue for a linear map $T:V\to V$, then V_λ is a subspace of V.

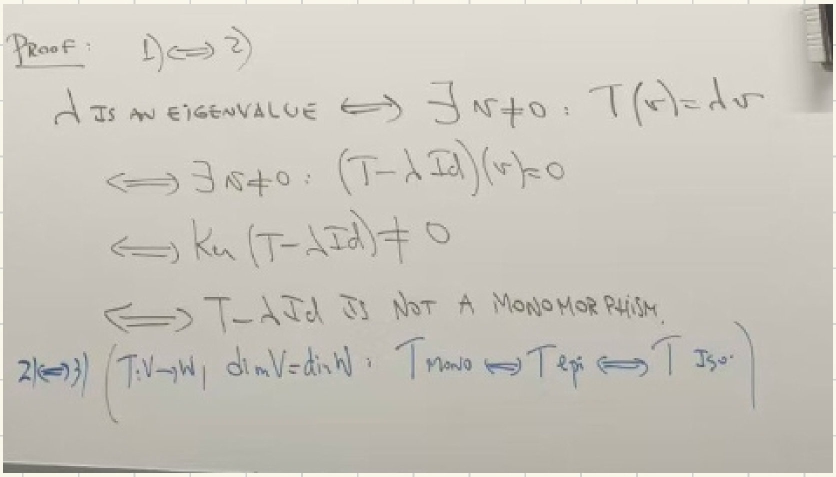
Proof:

Option 1: $0\in V_\lambda$. $v_1,v_2\in V_\lambda\Rightarrow v_1+v_2\in V_\lambda$. $\forall a\in F$. st $av\in V_\lambda$ $\forall v\in V_\lambda$.

Option 2: $V_\lambda=\{v: T(v)=\lambda v\}\leq V$. $=\{v: T(v)=\lambda\cdot Id v\}=\{v:(T-\lambda Id)(v)=0\}=\ker(T-\lambda Id)$. is a subspace.

THEO: let $T\in Hom_F(V,V)$. the following are equal:

1) $\lambda\in F$ is an Eigenvalue.
2) $T-\lambda Id$ is not Mono.
3) $T-\lambda Id$ is not Iso.
4) $[T-\lambda Id]_B$ is not invertible $\forall B$.
5) $[T-\lambda Id]_B$ is not invertible for some B.
6) $Det[T-\lambda Id]_B=0$ for some basis B.



PROOF: 1) ⟺ 2)

λ IS AN EIGENVALUE ⟺ $\exists v\neq0: T(v)=\lambda v$

⟺ $\exists v\neq0: (T-\lambda Id)(v)=0$

⟺ $\ker(T-\lambda Id)\neq0$

⟺ $T-\lambda Id$ IS NOT A MONOMORPHISM.

2) ⟺ 3) ($T:V\to W$, $\dim V=\dim W$: T mono ⟺ T epi ⟺ T Iso.)

LEMMA: 会考.

if $T \in Hom_F(V,V)$, B, B' are basis of V, $\Rightarrow \det[T]_B = \det[T]_{B'}$.

Proof:

Since $[T]_{B'} = [B]_{B'} \cdot [T]_B [B']_B = [B']_B^{-1} \cdot [T]_B \cdot [B']_B$.

$\Rightarrow \det[T]_{B'} = \det\left([B']_B^{-1} \cdot [T]_B \cdot [B']_B\right) = \det([B']_B^{-1}) \cdot \det[T]_B \cdot \det[B']_B$. ①

if $C \cdot C^{-1} = Id$ $C \in M_n(F)$ $\Rightarrow 1 = \det(Id) = \det(C \cdot C^{-1}) = \det C \cdot \det C^{-1} \Rightarrow \det C = 1/\det C^{-1} \in F$.

then ① $= (\det[B']_B)^{-1} \cdot \det[T]_B \cdot \det[B']_B = \det[T]_B\left[(\det[B']_B)^{-1} \cdot \det[B']_B\right] = \det[T]_B$.

So $\det[T]_{B'} = \det[T]_B$ $\forall B, B'$.

Back to the theo. take $T - \lambda Id : V \longrightarrow V$. B, B' basis of V. $\Rightarrow \det[T-\lambda Id]_B = \det[T-\lambda Id]_{B'}$.

Def:

let $T \in Hom_F(V,V)$ The characteristic polynomial associated to T is given by:

$P_T(x) = \det[xId - T]_B$ for any B basis for V.

$(\det[T-xId]_B = (-1)^n \det[xId - T]_B)$.

THEO:

λ is an Eigenvalue of $T \Leftrightarrow \lambda$ is a root of $P_T(x)$.

Proof: λ root of $P_T(x) \Leftrightarrow 0 = P_T(\lambda) = \det[\lambda Id - T]_B = (-1)^n \det[T - \lambda Id]_B \overset{(1)\Rightarrow(6)}{\Leftrightarrow} \lambda$ is Eigenvalue.

Example:

(1) $T(x,y) = (x+y, x+y)$. $[T]_B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

$P_T(x) = |[xId - T]_B| = |x[Id]_B - [T]_B| = \left| x \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right| = \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} = (x-1)^2 - 1 = x(x-2)$.

Eigenvalue for T: $x=0$ or $x=2$. (x is a root of $P_T(x)$)

$V_0 = \{v : T(v) = 0 \cdot v\} = \ker T = \{(x,y) \in F^2 : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\} = \{(x,y) : x+y=0\} = \{(x,-x) : x \in F\}$.

$V_2 = \{v : T(v) = 2v\} = \{v : (T - 2Id)(v) = 0\} = \ker(T - 2Id) = \{(x,y) \in F^2 : \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$
$= \{(x,y) : x-y=0\} = \{(x,x), x \in F\}$.

2). $[T]_B = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}$, $P_T(x) = \begin{vmatrix} x-3 & -1 & 1 \\ -2 & x-2 & 1 \\ -2 & -2 & x \end{vmatrix} = (x-1)(x-2)^2$. $\Rightarrow$ Eigenvalues: 1, 2.

$V_1 = \ker(T - 1d) = \{(x,y,z) : \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\} = \{(x,0,2x) : x \in F\} = <(1,0,2)>$.

$V_2 = \ker(T - 2Id) = \{(x,y,z) : \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\} = \{(x,x,2x), x \in F\} = <(1,1,2)>$.

Finally, example 1 is Diagonalizable: $B = \{(1,-1), (1,-1)\} \Rightarrow [T]_B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

example 2 is not Diagonalizable:

Eigenvectors: $(x,x,2x)$, $x \neq 0$. $\rightarrow$ These is not basis of Eigenvectors:

$(x,0,2x)$, $x \neq 0$. Jordan form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

6.4. CLASS 24.

$T: V \to V$, find B : $[T]_B$ is "EASY", for instance, $[T]_B = \text{diagonal} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

λ eigenvalue of $T \Leftrightarrow \exists v$ eigenvector for λ : $v \neq 0$ and $T(v) = \lambda v$.

How can we find eigenvalues?

$P_T(x) = \det([xId - T]_B)$, $\forall B$ basis : λ eigenvalue of $T \Leftrightarrow P_T(\lambda) = 0$.

eigenvector?

$V_\lambda = \{0\} \cup \{\text{eigenvectors of } \lambda\} = \{v \in V : T(v) = \lambda v\}$ = subspace of $\{v \in V : (T - \lambda Id)(v) = 0\} = \ker(T - \lambda Id)$.

(λ eigenvalue $\Rightarrow V_\lambda \neq \{0\} \Rightarrow \dim V_\lambda \geq 1$)

Def:

A LM $T: V \to V$ is Diagonalizable if $\exists B$ basis of V: $[T]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Remark:

1) T is diagonalizable $\Leftrightarrow \exists \lambda_1 \dots \lambda_n \in F$, $B = \{v_1, v_2 \dots v_n\}$: $T(v_i) = \lambda_i v_i \quad \forall i = 1, 2, \dots n$

$\Leftrightarrow \exists B$ a basis of eigenvectors.

2) If T is diagonalizable $\Rightarrow P_T(x) = |[xId - T]_B| = |xId - [T]_B| = \begin{vmatrix} x-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & x-\lambda_n \end{vmatrix} = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_n)$.

$\Rightarrow P_T(x)$ has all its roots in F.

Example.

1) $[T]_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P_T(x) = \begin{vmatrix} x-1 & -1 \\ 0 & x-1 \end{vmatrix} = (x-1)^2 \Rightarrow \lambda = 1$ is the unique eigenvalue.

$V_1 = \{(x,y) : T(x,y) = (x,y)\} = \{(x,y) : \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}\} = \{(x,y) : \begin{cases} x+y = x \\ y = y \end{cases}\} = \{(x,0), x \in F\} = <(1,0)>$.

eigenvectors $= \{(x,0), x \neq 0\}$, $\nexists B$ basis of eigenvectors.

$\Rightarrow T$ is not diagonalizable.

2) $T(x,y) = (x+y, x+y)$, $[T]_B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. $P_T(x) = x(x-2)$.

$V_0 = <(1,-1)>$, $V_2 = <(1,1)>$. $\Rightarrow B = \{(1,-1), (1,1)\}$ basis and $[T]_B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

3) $[T]_B = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, $P_T(x) = (x-1)(x-2)^2$. $\Rightarrow V_1 = <(1,0,2)>$ $V_2 = <(1,1,2)>$.

T is not diagonalizable since $\nexists B$ basis of eigenvectors.

4) $T(x,y) = (-y, x)$, $[T]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $P_T(x) = \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2 + 1$.

$T: R^2 \to R^2 \Rightarrow T$ is not diagonalizable since it has no eigenvalues.

$T: C^2 \to C^2 \Rightarrow i, -i$ are eigenvalues.

$V_i = \{(x,y) : T(x,y) = i(x,y)\} = \{(x,y) : (-y,x) = (ix, iy)\} = \{(x,y) : \begin{cases} y = -ix \\ x = iy \end{cases}\} = \{(iy, y), y \in C\} = <(i,1)>$.

$V_{-i} = \{(x,y) : (-y,x) = (-ix, -iy)\} = \{(x,y) : \begin{cases} y = ix \\ x = -iy \end{cases}\} = \{(-iy, y), y \in C\} = <(-i,1)>$.

$B = \{(i,1), (-i,1)\}$ is a basis of C^2. $\Rightarrow [T]_B = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

PROBLEM: Find conditions on T st T is diagonalizable.

Remark: eigenvectors associated to different eigenvalues are LI.

Proof: $v_1, v_2 \dots v_m$: $T(v_i) = \lambda_i v_i \quad v_i \neq 0$, $\lambda_i \neq \lambda_j \quad \forall i \neq j$. by induction on m:

$m = 1$: $a_1 v_1 = 0$, $v_1 \neq 0 \Rightarrow a_1 = 0$.

$m = 2$: $T(v_1) = \lambda_1 v_1$ $T(v_2) = \lambda_2 v_2$ $\lambda_1 \neq \lambda_2$

$a_1v_1 + a_2v_2 = 0 \Rightarrow 0 = T(0) = T(a_1v_1 + a_2v_2) = \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2 = \lambda_1 \cdot 0 = \lambda_1 a_1 v_1 + \lambda_1 a_2 v_2$.

$\Rightarrow \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2 = \lambda_1 a_1 v_1 + \lambda_1 a_2 v_2 \Rightarrow (\lambda_1 - \lambda_2) a_2 v_2 = 0$. since $\lambda_2 \neq \lambda_1$, $v_2 \neq 0 \Rightarrow a_2 = 0 \Rightarrow a_1 = 0$.

Suppose $m-1$ $\Rightarrow$ prove m is same.

LEMMA:

if $T \in Hom_F(V,V)$, λ is eigenvalue, $v \neq 0$, $T(v) = \lambda v$, $P(x) = a_n x^n + \cdots + a_1 x + a_0 \in F[x]$.

$\Rightarrow P(\lambda)$ is an eigenvalue of the linear map $P(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0 . Id : V \to V$ with eigenvector v.

Proof:

By induction we can prove that $T^k(v) = \lambda^k v$.

$k=1$. $T(v) = \lambda v$.

$k=2$. $T^2(v) = T(T(v)) = T(\lambda v) = \lambda \cdot T(v) = \lambda^2 \cdot v$.

Assume $T^{k-1}(v) = \lambda^{k-1} v \Rightarrow T^k(v) = T^{k-1}(T(v)) = T^{k-1}(\lambda v) = \lambda \cdot T^{k-1}(v) = \lambda^k \cdot v$

$(a_n T^n + \cdots + a_1 T + a_0 Id)(v) = a_n T^n(v) + a_{n-1} T^{n-1}(v) + \cdots + a_1 T(v) + a_0 v$

$= a_n \lambda^n \cdot v + a_{n-1} \lambda^{n-1} \cdot v + \cdots + a_1 \lambda \cdot v + a_0 v = P(\lambda) \cdot v$.

THEO: let $T \in Hom_F(V,V)$, let $\lambda_1 \ldots \lambda_n$ be eigenvalues on T, $\lambda_i \neq \lambda_j$. $\forall i \neq j$.

1) $W = V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_n} = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n}$.

2) if B_i is a basis of $V_{\lambda_i} \Rightarrow B_1 \cup B_2 \cup \cdots \cup B_n$ is a basis of W.

3) $\dim W = \dim V_{\lambda_1} + \dim V_{\lambda_2} + \cdots + \dim V_{\lambda_n}$

Proof:

We know that $S_1 + S_2 + \cdots + S_m = S_1 \oplus S_2 \oplus \cdots \oplus S_n \Leftrightarrow$ any $v \in S_1 + S_2 + \cdots + S_n$ can be written in unique way.

As $v = v_1 + v_2 + \cdots + v_n$, $v_i \in S_i \Leftrightarrow 0 = v_1 + v_2 + \cdots + v_n$ $v_i \in S_i \Rightarrow v_1 + v_2 + \cdots v_n = 0$.

Let $v_1 + v_2 + \cdots + v_n = 0$. $v_i \in V_{\lambda_i}$ that is $T(v_i) = \lambda_i v_i$ We have to prove that $v_i = 0$ $\forall i$.

Set $P_i(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_{i-1})(x-\lambda_i+1)(x-\lambda_{i+1})\cdots(x-\lambda_n) \in F[x]$ $\lambda_i \neq \lambda_j$. $\forall j \neq i$.

$P_i(\lambda_i) = 1$.

$P_j(\lambda_j) = 0$.

$$0 = \underset{\text{LM}}{\underset{\Downarrow}{P_i(T)}}(\underset{=0}{\underset{\Downarrow}{v_1 + v_2 + \cdots + v_n}}) = \underset{0}{\underset{\shortparallel}{P_1(\lambda_1)}} v_1 + \underset{0}{\underset{\shortparallel}{P_2(\lambda_2)}} v_2 + \cdots + \underset{1}{\underset{\shortparallel}{P_i(\lambda_i)}} v_i + \cdots + \underset{0}{\underset{\shortparallel}{P_n(\lambda_n)}} v_n = v_i \quad \forall i.$$

THEO: let $T \in Hom_F(V,V)$ the FAE:

1) T is diagonalizable.

2) $P_T(x) = (x-\lambda_1)^{c_1}(x-\lambda_2)^{c_2} \cdots (x-\lambda_m)^{c_m}$, $\lambda_i \neq \lambda_j$ $\forall i \neq j$ $(n = \dim V = \dim P_T(x) = c_1 + c_2 + \cdots + c_m)$.

and $\dim V_{\lambda_i} = c_i$ $\forall i = 1, 2 \ldots m$.

3) $\dim V = \dim V_{\lambda_1} + \dim V_{\lambda_2} + \cdots + \dim V_{\lambda_m}$

4) $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}$.

Proof:

1) $\Rightarrow$ 2):

$$T \text{ diag} \Rightarrow \exists B \text{ basis } [T]_B = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_m & \\ & & & & \lambda_m \end{bmatrix} \begin{matrix} c_1 \\ \\ c_m \\ \\ \\ \end{matrix} \Rightarrow P_T(x) = (x-\lambda_1)^{c_1} \cdots (x-\lambda_m)^{c_m}.$$

Moreover, $\dim V_{\lambda_i} \geq c_i$ since the basis B contains c_i LI. eigenvectors associated to λ_i.

We know $V \geq V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m} \Rightarrow n = \dim V \geq \dim V_{\lambda_1} + \dim V_{\lambda_2} + \cdots + \dim V_{\lambda_m} \geq c_1 + c_2 + \cdots + c_m = n$.

if $\dim V_{\lambda_i} > c_i \Rightarrow n \geq \dim V_{\lambda_1} + \dim V_{\lambda_2} + \cdots + \dim V_{\lambda_m} \geq c_1 + c_2 + \cdots + c_m > n \Rightarrow n > n$ Contradiction.

$\Rightarrow \dim V_{\lambda_i} = c_i$.

2) → 3):

$n = \deg P_T(x) = c_1 + c_2 + \cdots + c_m$. $\Rightarrow \dim V = n = c_1 + c_2 + \cdots + c_m = \dim V_{\lambda_1} + \dim V_{\lambda_2} + \cdots + \dim V_{\lambda_m}$.

3) → 4):

We know that $V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m} \subseteq V$.

and $\dim(V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}) = \dim V_{\lambda_1} + \cdots + \dim V_{\lambda_m} = \dim V = n$. $\Rightarrow V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m} = V$.

4) → 1):

if $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_m} \Rightarrow \exists B$ a basis of B st $B = B_1 \cup B_2 \cup B_3 \cdots \cup B_m$. B_i basis of V_{λ_i}.

B_i = basis of eigenvectors associated to λ_i.

$\Rightarrow B$ is a basis of eigenvectors $\Rightarrow T$ is diagonalizable.

Corollary: if $T \in \mathrm{Hom}_F(V,V)$, $P_T(x) = (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n) \in F[x]$, $\lambda_i \neq \lambda_j$ $\forall i \neq j \Rightarrow T$ is diagonalizable.

Proof: We know that $\dim V_{\lambda_i} \geq 1$.

Moreover, $V \supseteq V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n} \Rightarrow n = \dim V \geq \dim V_{\lambda_1} + \cdots + \dim V_{\lambda_n} \geq 1 + 1 + \cdots + 1 = n \Rightarrow \dim V_{\lambda_i} = 1$ $\forall i$.

Def: λ eigenvalue of T.

Algebraic multiplicity of λ = Mutiplicity of the root λ in $P_T(x)$.

Geometric multiplicity of λ = $\dim V_\lambda$.

THEO: T is diagonalizable $\Leftrightarrow P_T(x)$ has all its roots in F and geom-multi(λ) = alg-mult(λ), $\forall \lambda : P_T(\lambda) = 0$.

Remark:

1) Geom-mult$(\lambda) \geq 1$ and Alg-mult$(\lambda) \geq 1$ $\forall \lambda$ root of $P_T(x)$.

2) for any eigenvalue λ, geo-mult$(\lambda) \leq$ alg-mult(λ).

Proof:

(2) let k = geo-mult(λ). $\Rightarrow \{v_1 \ldots v_k\}$ basis of V_λ.

$\{v_1 \ldots v_k\}$ LI in V. we can extend it to a basis $B = \{v_1, v_2 \ldots v_k, v_{k+1} \ldots v_n\}$

$$[T]_B = \begin{bmatrix} \lambda & 0 & \cdots & 0 & a_{1k+1} & \cdots & a_{1n} \\ \vdots & \lambda & & & \vdots & & \vdots \\ \vdots & & \ddots & \lambda & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_{nk+1} & \cdots & a_{nn} \end{bmatrix} \Rightarrow P_T(x) = \begin{vmatrix} x-\lambda & 0 & -a_{1k+1} & \cdots \\ 0 & x-\lambda & & \vdots \\ \vdots & & x-a_{k+1k+1} & \vdots \\ 0 & 0 & -a_{nk+1} & \cdots \quad x-a_{nn} \end{vmatrix} = (x-\lambda)(x-\lambda)\ldots(x-\lambda) \cdot \begin{vmatrix} x-a_{k+1k+1} & & \\ \vdots & \ddots & \\ -a_{nk+1} & \cdots & x-a_{nn} \end{vmatrix} = (x-\lambda)^k \cdot q(x).$$

$\Rightarrow$ alg-mult$(\lambda) \geq k$.

$\mathrm{Hom}_F(V,V) \xrightarrow{\simeq} M_n(F)$.

$T \longleftrightarrow [T]_B = A$.

We can define: $\underset{\lambda}{\underset{\downarrow}{\text{eigenvalues}}}$ and $\underset{0 \neq [x] \in M_n(F)}{\underset{\downarrow}{\text{eigenvectors}}}$ for A.

$A \cdot [x] = \lambda \cdot [x]$

A diagonalizable $\Leftrightarrow \exists C \in M_n(F)$ invertable st $C^{-1} \cdot A \cdot C = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\updownarrow$

$[T]_{B_1} \overset{\text{Diag}}{\Longleftrightarrow} [T]_{B_2} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $[T]_{B_2} = [B_1]_{B_2} \cdot [T]_{B_1} \cdot [B_2]_{B_1}$

6.9. CIASS 25.

$\mathrm{Hom}_F(V.V) \longrightarrow M_n(F)$

$T \to [T]_B$. $\det T = \det[T]_B$. we know $\det[T]_B = \det[T]_{B'}$ $\forall B.B'$ basis.

$P_T(x) = \det[xId - T]_B = \det(xId - T)$.

$$\begin{tikzcd}
\text{T diagonalizable} \arrow[r] \arrow[d, Leftrightarrow] & \text{A diagonalizable} \arrow[d, Leftrightarrow] \\
{[T]_{B''} = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}} & {\exists C \text{ inverible. } C^{-1}.A.C = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}.}
\end{tikzcd}$$

$: A = [T]_B \; : D = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} = [T]_{B''} = [B']_B^{-1}\,[T]_B . [B]_{B'}$.

Def:

Given $A.B \in M_n(F)$, we say that A is similar to B if $\exists C \in M_n(F)$ inverse: $A = C^{-1}.B.C$.

Notation: $A \sim B$.

Proposition: "Similar" is an Equivalence Relation. $A \sim B \Leftrightarrow \exists C.\ A = C^{-1}.B.C$.

Proof:

① $A \sim A. \Leftrightarrow \exists Id:\ A = Id^{-1}.A.Id$.

② $A \sim B \Leftrightarrow \exists C,\ A = C^{-1}.B.C \Leftrightarrow \exists C^{-1}:\ B = (C^{-1})^{-1}.A.C^{-1} \Leftrightarrow B \sim A$.

③ $A_1 \sim A_2$. $A_2 \sim A_3 \Leftrightarrow \exists C_1 C_2:\ A_1 = C_1^{-1}.A_2 C_1$ $A_2 = C_2^{-1}.A_3.C_2 \Rightarrow A_1 = C_1^{-1}.(C_2^{-1}.A_3.C_2)C_1 = (C_2C_1)^{-1}.A_3(C_2C_1). \Rightarrow A_1 \sim A_3$.

THEO:

$A_1 \sim A_2 \Leftrightarrow A_1$ and A_2 are associated to the same the LM: $T: F^n \longrightarrow F^n \Leftrightarrow \exists T: F^n \longrightarrow F^n:\ [T]_{B_1} = A_1.\ [T]_{B_2} = A_2.\ B_1, B_2$ basis

Proof:

$\Leftarrow$) Assume $[T]_{B_1} = A_1$ $[T]_{B_2} = A_2$. then $A_1 = [T]_{B_1} = [B_2]_{B_1}[T]_{B_2}.[B_1]_{B_2} = [B_1]_{B_2}^{-1}.A_2.[B_1]_{B_2} \Rightarrow A_1 \sim A_2$.

$\Rightarrow$) Assume $A_1 \sim A_2$ define $T: F^n \to F^n$ st $A_2 = [T]_{B_2}$ for some basis B_2 of F^n.

$A_1 \sim A_2 \Rightarrow \exists C$ inverse: $A_1 = C^{-1}.A_2.C = C^{-1}.[T]_{B_2}.C = [B_1]_{B_2}^{-1}.[T]_{B_2}.[B_1]_{B_2} = [T]_{B_1}$. where B_1 is the basis st $[B_1]_{B_2} = C$

Example.

Find B_1 st $[B_1]_{B_2} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 2\\ 1 & 0 & 0\end{bmatrix}$, $B_2 = \{(1,1,1), (0,1,1), (0,0,1)\}$. $B_1 = \{v_1, v_2, v_3\}$.

$v_1 = 1.(1.1.1) + 0.(0.1.1) + 1.(0.0.1)$. $v_2 = 1.(1,1,1) + 1.(0,1,1) + 0.(0,0,1)$. $v_3 = 0(1,1,1) + 2.(0,1,1) + 0.(0,0,1)$.

Remark: A is diagonalizable $\overset{Def}{\Longleftrightarrow} \exists C: D = C^{-1}.A.C$.

$$D = \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} \overset{Def}{\Longleftrightarrow} A \sim \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} \overset{Theo}{\Longleftrightarrow} A = [T]_{B_1},\ \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} = [T]_{B_2} \overset{Def}{\Longleftrightarrow} T \text{ is diagonalizable.}$$

Proposition: $A_1 \sim A_2 \Rightarrow P_{A_1}(x) = P_{A_2}(x)$, where $P_A(x) = \det(xId - A)$.

Proof:

$A_1 \sim A_2 \overset{Theo}{\Longrightarrow} \exists T: F^n \to F^n:\ A_1 = [T]_{B_1},\ A_2 = [T]_{B_2}$.

$P_{A_i}(x) = \det(\underset{\substack{\swarrow \\ Id \in M_n(F).}}{xId} - A_i) = \det([\underset{\substack{\swarrow \\ Id: F^n \to F^n.}}{xId} - T]_{B_i}) = P_T(x)$. $Id = [Id]_{B_i}$

$\Rightarrow P_{A_1}(x) = P_{A_2}(x) = P_T(x)$.

Remark: $\not\Leftarrow$. $P_{A_1}(x) = P_{A_2}(x) \not\Rightarrow A_1 \sim A_2$.

Counterexample

$A_1 = Id$. $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $P_{A_1}(x) = \begin{bmatrix} x-1 & 0 \\ 0 & x-1 \end{bmatrix} = (x-1)^2$. $P_{A_2}(x) = \begin{vmatrix} x-1 & -1 \\ 0 & x-1 \end{vmatrix} = (x-1)^2$.

$A_1 \sim A_2 \Leftrightarrow \exists C$ inverse : $A_2 = C^{-1} A_1 C = C^{-1} Id. C = Id$. but $A_2 \neq Id$. $\Rightarrow$ Contradiction.

THEO: if $A_1 A_2$ are diagonalizable. then $A_1 \sim A_2 \Leftrightarrow P_{A_1}(x) = P_{A_2}(x)$.

Remark:

$\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \sim \begin{pmatrix} \lambda_2 & & 0 \\ & \lambda_3 & \\ 0 & & \lambda_1 \end{pmatrix}$ since if $\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} = [T]_B \Rightarrow \begin{pmatrix} \lambda_2 & & 0 \\ & \lambda_3 & \\ 0 & & \lambda_1 \end{pmatrix} = [T]_{B'}$.

$B_1 = \{v_1, v_2, v_3\}$. $B_2 = \{v_2, v_3, v_1\}$.

Prove the theo:

$\Rightarrow$) Previous proposition.

$\Leftarrow$) $P_{A_1}(x) = P_{A_2}(x)$. $C_1^{-1} A_1 C_1 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $C_2^{-1} A_2 C_2 = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$

$P_{C_i^{-1} A_i C_i}(x) = \det(xId - C_i^{-1} A_i C_i) = \det(x C_i^{-1} Id C_i - C_i^{-1} A_i C_i) = \det(C_i^{-1}(xId - A_i) C_i)$.

$= \det C_i^{-1} \cdot \det(xId - A_i) \cdot \det C_i = \det(xId - A_i) = P_{A_i}(x)$.

$\Rightarrow \begin{vmatrix} x-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & x-\lambda_n \end{vmatrix} = \begin{vmatrix} x-\mu_1 & & 0 \\ & \ddots & \\ 0 & & x-\mu_n \end{vmatrix} \Rightarrow (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n) = (x-\mu_1)(x-\mu_2)\ldots(x-\mu_n). \Rightarrow \{\lambda_1 \ldots \lambda_n\} = \{\mu_1 \cdots \mu_n\}$.

THEO: let $T: V \to V$. st $P_T(x)$ has all its roots in F. $P_T(x) = (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n)$. $\lambda_i \in F$.

Then $\exists B$ a basis of V st $[T]_B$ is upper triangular, that is $[T]_B = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Proof:

By induction on $n = \dim V$.

$n=1$. $D_T(x) = (x-\lambda_1) \Rightarrow [T]_B = [\lambda_1] \Rightarrow T = \lambda_1 Id$.

Assume true for $k-1$.

λ_1 is the root of $P_T(x) \Rightarrow \lambda_1$ is eigenvalue $\Rightarrow \exists v_1 \neq 0$, $T(v_1) = \lambda_1 v_1$.

Extend $\{v_1\}$ to a basis $B = \{v_1, v_2 \ldots v_k\} \Rightarrow [T]_B = \begin{bmatrix} \lambda_1 & a_{12} & a_{13} & \cdots & a_{1k} \\ 0 & a_{22} & & & \\ \vdots & \vdots & & & \\ 0 & a_{k2} & \cdots & \cdots & a_{kk} \end{bmatrix}$

Let $W = \operatorname{span}(v_2, v_3 \cdots v_k)$. $\Rightarrow \dim W = k-1$.

$T': W \to W$: $[T']_{B'} = \begin{bmatrix} a_{22} & \cdots & a_{2k} \\ \vdots & & \\ a_{k2} & \cdots & a_{kk} \end{bmatrix}$ for $B' = \{v_2 \ldots v_k\}$.

$T(v_2) = a_{12} v_1 + \underbrace{a_{22} v_2 + a_{32} v_2 + \cdots + a_{k2} v_k}_{T'(v_2)} = a_{12} \cdot v_1 + T'(v_2)$.

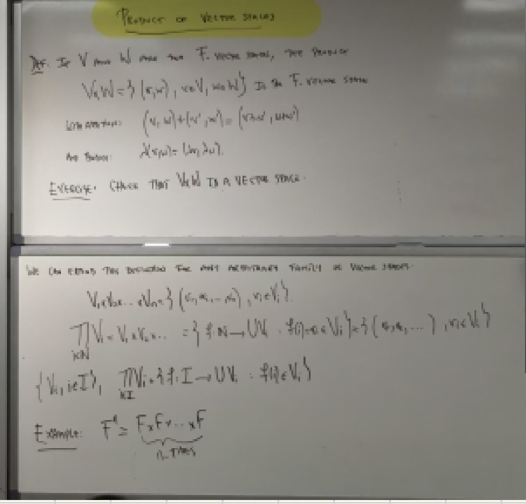
$\vdots$

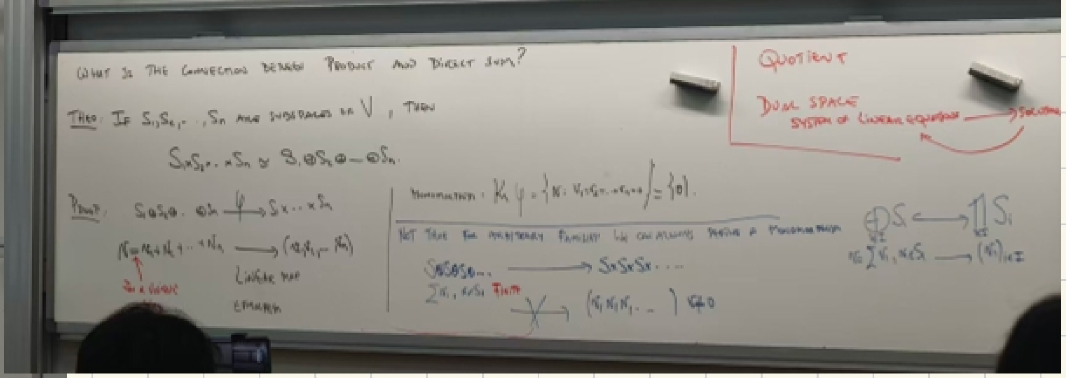
$T(v_k) = a_k v_1 + T'(v_k)$.

let $\bar{B} = \{v_1, w_1 \ldots w_n\}$. $[T]_{\bar{B}} = \begin{bmatrix} \lambda_1 & c_2 & c_3 & \cdots & c_k \\ 0 & \lambda_2 & c_{23} & \cdots & c_{2k} \\ \vdots & 0 & \lambda_3 & \cdots & \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix}$ since $[T']_{B''} = \begin{bmatrix} \lambda_2 & c_{23} & \cdots & c_{2k} \\ & \lambda_3 & & \vdots \\ 0 & & \ddots & \\ & & & \lambda_k \end{bmatrix}$ because T' is in $k-1$.

$T(w_i) = T(b_2 v_2 + \cdots + b_n v_n) = b_2 T(v_2) + \cdots + b_k T(v_k) = b_2(a_{12} v_1 + T'(v_2)) + \cdots + b_k(a_{1k} v_1 + T'(v_k)) = (b_2 a_{12} + \cdots + b_k a_{1k}) v_1 + T'(b_2 v_2 + \cdots + b_k v_k) = c_i v_1 + T'(w_i)$.

▨



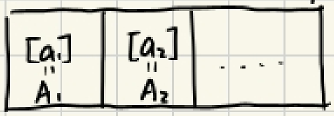


6.11. ClASS 26.

" ~ " **EQUIVAlENCE RELATION IN A.** $A/\sim$ = Quotient Set = $\{[a], a\in A\} = \{A_i, i\in I\}$

$[a] = \{b\in A : b\sim a\}$

set A. →



V a vector space, $S\subseteq V$ subspace:

$v\sim w \Leftrightarrow v-w\in S.$

EQUIVAlENT RELATION:

1) $v\sim v \Leftrightarrow v-v=0\in S.$

2) $(v\sim w \Rightarrow w\sim v) \Leftrightarrow (v-w=u\in S, \Rightarrow w-v=-u=(-1).u\in S)$ ✓.

3) $(v\sim w, w\sim r \Rightarrow r\sim r) \Leftrightarrow (v-w\in S, w-r\in S \Rightarrow v-u = v-w+w-u\in S)$ ✓.

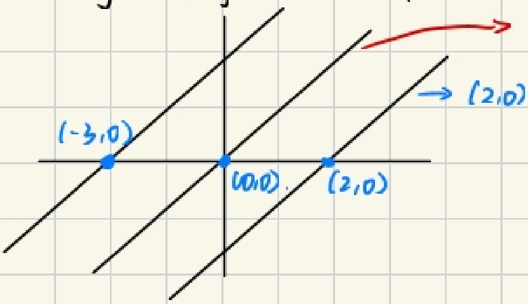
V/S = Quotient set = $\{[v], v\in V\}$, $[v] = \{w\in V: w\sim v\} = \{w\in V: w-v\in S\}$

$= \{w\in V: w-v=s, s\in S\}.= \{w\in V: w=v+s, s\in S\} = v+S\subseteq V.$

Example.

$V=\mathbb{R}^2$, $S=\langle(1,1)\rangle = \{(a,a), a\in\mathbb{R}\}$: $(x',y')\sim(x,y) \Leftrightarrow (x'-x, y'-y)=(a,a), a\in\mathbb{R} \Leftrightarrow (x,y)+(a,a)=(x',y').$

$\Rightarrow [(x,y)] = \{(x,y)+(a,a), a\in\mathbb{R}\}.$ let $(x,y)=(-3,0)$. $\Rightarrow$ $(-3,0)+S = [(-3,0)]$.



$S=(0,0)+S=[(0,0)]$

$\rightarrow (2,0)+S = \{(2,0)+\lambda(1,1)\}.$

V/S = Set of all lines parallel to S: $x=y$.

$= \{[x_0,y_0] = (x_0,y_0)+S = [(x_0-y_0,0)]\}$

Since $(x_0,y_0)\sim(x_0-y_0,0) \Rightarrow (x_0,y_0)-(x_0-y_0,0) = (y_0,y_0)\in S.$

$V/S \longleftrightarrow \mathbb{R}.$

$[(x_0,y_0)] \longrightarrow x_0-y_0.$

THEO: if S is a subspace of V, then V/S is a vector space.

(1) $[v]+[w] = [v+w].$

(2) $\lambda[v] = [\lambda v]$

Proof:

(1). $v \sim v'$. $w \sim w'$ $\Rightarrow$ $v - v' \in S$. $w - w' \in S$ $\Rightarrow$ $v + w - (v' + w') \in S$.

$[v] = [v']$ $[w] = [w']$ $\Rightarrow$ $[v+w] = [v'+w']$.

(2) $v \sim v' \Rightarrow v - v' \in S. \Rightarrow \lambda . v - \lambda v' = \lambda . (v - v') \in S. \Rightarrow \lambda v \sim \lambda v'$.

$[v] = [v'] \Rightarrow [\lambda v] \sim [\lambda v']$.

$[v] + [w] = [v+w] = [w] + [v]$.

APPLICATION:

if $T: V \to W$ not a Mono. we can transform it into a Mono Moreover, into an Iso.

THEO:

let $T: V \to W$, be a LM. $\ker T \subseteq V$. subspace. Then $\exists! \, T': V/\ker T \longrightarrow W$ a Mono

have the diagram:

$$\begin{tikzcd} v \arrow[d, "\text{Epi.}"'] \\ {[v]} \end{tikzcd}$$

delete the vectors which we don't want.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & W \arrow[dl, leftrightarrow, "T'"] \\ V/\ker T & \end{tikzcd}$$

$V/\ker T = \{[v] : v - v' = v_0, v_0 \in \ker T\}$.

$\ker T = 0 + \ker T = [0]$.

Proof of the theo:

Since $[v] = v + \ker T. \Rightarrow T'([v]) = T'(v) + T'([0]) = T'(v) \Rightarrow T'([v]) = T'(v)$. and we define $T'([v]) = T(v)$.

Well-defined: $v \sim v' \Rightarrow v - v' \in \ker T \Rightarrow T(v) = T(v')$

$[v] = [v'] \Rightarrow T'([v]) = T'([v'])$.

T' is LM.

$\Rightarrow \tilde{T}: V/\ker T \to \operatorname{Im} T$. Iso. $\tilde{T}([v]) = T'([v]) = T(v)$.

$$\begin{tikzcd} R^2 \arrow[r, "T"] \arrow[d] & R \arrow[dl, leftrightarrow, "\text{Iso.}"] \\ R^2/S & \end{tikzcd}$$

$T(x, y) = x - y$. $\ker T = S$.

THEO: if $\dim V < \infty$. then $\dim V/S = \dim V - \dim S$.

Proof: define $\pi: V \longrightarrow V/S$. Epi. $\dim V = \dim \ker \pi + \dim \operatorname{Im} \pi$.

$v \mapsto [v]$. $= \dim S + \dim V/S$. ∎

$\ker \pi = \{v : [v] = [0]\} = \{v. \, v - 0 \in S\} = S$.

DUALITY.

Homogeneous Systems of Linear Equations in n unknowns. $\longrightarrow$ S subspace of solutions, $S \subseteq F^n$.

Def: if V is a vector space, the Dual of V is the vector space $V^* = \{T: V \to F\} = \operatorname{Hom}_F(V, F)$.

THEO: $\dim V^* = \dim V$. // **Proof:** $V^* = \operatorname{Hom}_F(V, F) \overset{\sim}{\longleftrightarrow} M_{1 \times n}(F)$. $\operatorname{Hom}(V, W) \overset{\sim}{\longleftrightarrow} M_{m \times n}(F)$.

$\dim V = n = \dim V^*$. $\dim V = n$. $\dim W = m$.

Def: if $B = \{v_1, \dots v_n\}$ is a basis of V, then $B^* = \{v_1^*, v_2^* \dots v_n^*$, $v_i^* = V \longrightarrow F$: $\begin{cases} v_i^*(v_i) = 1. \\ v_i^*(v_j) = 0 \quad j \neq i. \end{cases}$ $\Rightarrow$ LM.

Example

1) $\mathcal{B} = \{(1,0),(0,1)\} = \{e_1, e_2\}$ ⇒ $\mathcal{B}^* = \{e_1^*, e_2^*\}$, $e_i^* = R^2 \to R$.

$e_1^*(x,y) = e_1^*(x\cdot e_1 + y\cdot e_2) = x\cdot e_1^*(e_1) + y\cdot e_1^*(e_2) = x\cdot 1 + y\cdot 0 = x$.

$e_2^*(x,y) = e_2^*(x\cdot e_1 + y\cdot e_2) = x\cdot e_2^*(e_1) + y\cdot e_2^*(e_2) = x\cdot 0 + y\cdot 1 = y$.

for any $T: R^2 \to R$. $T(e_1) = T(1,0) = a$ ⇒ $T(x,y) = x\cdot T(e_1) + y\cdot T(e_2)$ ⇒ $T = ae_1^* + be_2^*$.
$T(e_2) = T(0,1) = b$. $= ax + by$

THEO: if $\mathcal{B}$ is a basis for V then $\mathcal{B}^*$ is a basis of V^*.

Proof:

ntp. $\{v_1^*, v_2^*, \dots v_n^*\}$ is LI.

$0 = a_1 v_1^* + a_2 v_2^* + \dots + a_n v_n^* : V \to F$.

$$0(v_i) = 0 = (a_1 v_1^* + \dots + a_n v_n^*)(v_i) = a_1 \underbrace{v_1^*(v_i)}_{0} + a_2 \underbrace{v_2^*(v_i)}_{0} + \dots + a_i \underbrace{v_i^*(v_i)}_{1} + \dots + a_n \underbrace{v_n^*(v_i)}_{0} = a_i$$

⇒ for $\forall a_i = 0$. $i = 1, 2 \cdots n$. ⇒ $\mathcal{B}^*$ is LI.

Since $\dim V = \dim V^*$ ⇒ $\mathcal{B}^*$ is a basis.

Def: $T^*: W^* \longrightarrow V^*$

$(\varphi: W \to F) \longmapsto T^*(\varphi): V \to F$: $T^*(\varphi) = \varphi \circ T$.

$$\begin{tikzcd} V \arrow[rr] \arrow[dr, "T"'] & & F \\ & W \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We can check: T^* is LM.

ANNIHILATOR: X°.

Def: $X \subseteq V$ subset, $X^\circ = \{\varphi \in V^*, \varphi(x) = 0, \forall x \in X\}$. $\varphi: V \to F$.

Proposition: $X \subseteq V \Rightarrow X^\circ$ is a subspace of V^*.

Proposition: $S \subseteq V$ subspace. then $\dim S^\circ + \dim S = \dim V = \dim V^*$.

Proof of the second pro:

$T: S \longrightarrow V$ Inclusion, $T^*: V^* \longrightarrow S^*$, $\operatorname{Im} T^* = S^*$ $\ker T^* = S^\circ$.

$T(s) = s$. ⇒ $\dim V^* = \dim V = \dim \ker T^* + \dim \operatorname{Im} T^* = \dim S^\circ + \dim S^* = \dim S^\circ + \dim S$. ☒

THEO:

let $S \subseteq V$ subspace, $S^\circ = \{\varphi \in V^* : \varphi(x) = 0\ \forall x \in S\}$. $U = \{v \in V : \varphi(v) = 0\ \forall \varphi \in S^\circ\}$.

Then $S = U$.

Proof:

$S \subseteq U$: $x \in S \Rightarrow \varphi(x) = 0\ \forall \varphi \in S^\circ \Rightarrow x \in U$. $\Rightarrow S \subseteq U$.

$S \supseteq U$: by Contradiction, assume $\exists v \in U$, $v \notin S$. $(v \neq 0)$. take $\{v_1 \dots v_r\}$ a basis of S.

$v \notin S \Rightarrow \{v_1 \dots v_r, v\}$ LI. Extend to a basis of V: $\{v_1 \dots v_r, v, v_{r+2}, \dots v_n\}$

So $v^* \in V^*$, $v^*(v_1) = 0 \dots v^*(v_r) = 0 \Rightarrow v^*(x) = 0\ \forall x \in S \Rightarrow v^* \in S^\circ$ but $v^*(v) = 1 \Rightarrow v \notin U$.

⇒ Contradiction. $\Rightarrow \forall v \in U \Rightarrow v \in S \Rightarrow U \subseteq S$.

$\Rightarrow S = U$. ☒

Example.

$S \subseteq R^4$. $S = \text{span}\{(1,1,0,1), (-1,1,1,0)\} \subseteq R^4$.

$S^0 = \{\varphi = a_1e_1^* + a_2e_2^* + a_3e_3^* + a_4e_4^* : \varphi(1,1,0,1) = 0.\ \varphi(-1,1,1,0) = 0\}. = \{\varphi = a_1e_1^* + a_2e_2^* + a_3e_3^* + a_4e_4^* : \begin{matrix} -a_1 + a_2 + a_3 = 0 \\ a_1 + a_2 + a_4 = 0 \end{matrix}\}.$

$a_1 = -a_2 + a_4$

$a_3 = -2a_2 + a_4$.

let $a_2 = a$, $a_4 = b$.

$= \{(-a-b)e_1^* + ae_2^* + (-2a-b)e_3^* + be_4^*,\ a, b \in R\}$.

$= \text{span}\{-e_1^* + e_2^* - 2e_3^*,\ -e_1^* - e_3^* + e_4^*\}$.

$S = U = \{x = (x_1, x_2, x_3, x_4) : \begin{matrix} (-e_1^* + e_2^* - 2e_3^*)(x) = 0 \\ (-e_1^* + e_3^* + e_4^*)(x) = 0 \end{matrix}\} = \{(x_1, x_2, x_3, x_4) : \begin{matrix} -x_1 + x_2 - 2x_3 = 0 \\ -x_1 - x_3 + x_4 = 0 \end{matrix}\}$.