3.21.

① Let $V=\{f:\mathbb{R}\to\mathbb{R}.\ f$ is a function$\}$. real vector space. Decide if the following subset of V are subspaces.

a) $S=\{f\in V,\ f(0)=f(1)\}$.

0 is a constant function, always equal to 0.

Let's check the properties: $0\in S$ (0 function, $0(x)=0\ \forall x\in\mathbb{R}$). $0(0)=0=0(1)$.

- if $f,g\in S,\Rightarrow f(0)=f(1)$ and $g(0)=g(1)$. $\Rightarrow f+g\in S$. since $(f+g)(0)=f(0)+g(0)=f(1)+g(1)=(f+g)(1)$.
- if $f\in S,\ \alpha\in\mathbb{R}\Rightarrow f(1)=f(0)\Rightarrow \alpha f(1)=\alpha f(0)\Rightarrow(\alpha\cdot f)(1)=(\alpha f)(0)\Rightarrow \alpha f\in S$.

b) $S=\{f\in V,\ f(3)=1+f(-5)\}$.

This is not a vector space. Since $0\notin S$. $0(3)=0$ but $1+0(-5)=1+0=1\neq 0.\neq 0(3)$.

S is also not closed by addition.

$f(x)=\begin{cases}1 & \text{if } x=3.\\ 0 & \text{if } x\neq 3.\end{cases}\Rightarrow f\in S.\ (f(3)=1=1+f(-5)=1+0=1)$.

$g(x)=\begin{cases}1 & \text{if } x\neq 5\\ 0 & \text{if } x=5\end{cases}\Rightarrow g\in S\ (g(3)=1=1+g(-5)=1+0=1)$.

but $f+g\notin S$ since $\left.\begin{aligned}(f+g)(3)&=f(3)+g(3)=1+1=2.\\ 1+(f+g)(-5)&=1+f(-5)+g(-5)=1+0+0=1\end{aligned}\right\}\neq$.

c) $S=\{f\in V: f(x^2)=f^2(x)\}$.

Not a subspace. Let $f(x)=1.\in S.\ (f(x^2)=1=f^2(x))$. $g(x)=x\in S\ (g(x^2)=x^2=g^2(x))$.

$\Rightarrow(f+g)(x)=f(x)+g(x)=1+x$. $(f+g)(x^2)=f(x^2)+g(x^2)=1+x^2\neq[(f+g)(x)]^2.=(1+x)^2$. $\Rightarrow$ So $f+g$ is not in S.

② Prove the following:

a) The only subspaces of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$.

Let S be a subspace of $\mathbb{R}.\Rightarrow 0\in S$. We have two cases:

i) 0 is the only element of $S\Rightarrow S=\{0\}$ ✓.

ii) 0 is not the only element of S. Let's prove $S=\mathbb{R}$.

Since $S\neq\{0\}$. $\exists v\in S.\ v\neq 0$. We want to prove that $\mathbb{R}\subseteq S$. (We always have $S\subseteq\mathbb{R}$).

Let $a\in\mathbb{R}.\Rightarrow a=a\cdot 1=a\cdot(v^{-1}v)=(av^{-1})\cdot v$, $av^{-1}\in\mathbb{R}$. $v\in S$. Since $v\neq 0$, $\exists v^{-1}\in\mathbb{R}.\Rightarrow a\in S$. Since S is a subspace, $\alpha\cdot v\in S,\ \forall\alpha\in\mathbb{R},\ \forall v\in S$.

Therefore $\mathbb{R}\subseteq S\subseteq\mathbb{R}.\Rightarrow S=\mathbb{R}$.

b) The only subspace of $\mathbb{R}^2$ are $\{0\}$, $\mathbb{R}^2$ and $l=\{\lambda(a,b):\lambda\in\mathbb{R}\}$ for $(a,b)\in\mathbb{R}^2$.

Let S be a subspace of $\mathbb{R}^2$. we have 2 cases.

- 0 is the only one element of S. $\Rightarrow S=\{0\}$.
- 0 is not the only one element of S. $\Rightarrow\exists u=(a,b)\in S$. st $(a,b)\neq(0,0)$. Since S is a subspace $\Rightarrow\lambda(a,b)\in S\ \forall\lambda\in\mathbb{R}$.

So if we denote $L=\{\lambda(a,b).\ \lambda\in\mathbb{R}\}\Rightarrow L\subseteq S$. We have 2 cases:

If $L=S$, we have done.

If $L\subsetneq S$. let's prove $S=\mathbb{R}^2$. Since $L\subsetneq S\Rightarrow\exists v=(c,d)\in S$. but $\notin L$. So $(c,d)\neq\lambda(a,b)$ for any $\lambda\in\mathbb{R}$.

Therefore any $(x,y)\in\mathbb{R}^2$. can be writen as $(x,y)=\lambda(a,b)+\alpha(c,d)$ $\lambda\cdot(a,b)\in S,\ \alpha\cdot(c,d)\in S.\Rightarrow(x,y)\in S.\Rightarrow\mathbb{R}^2\subseteq S\subseteq\mathbb{R}^2\Rightarrow S=\mathbb{R}^2$

③ Let U,W be 2 subspaces of V (Since $U\cap W$ is a subspace of V). Prove that $U\cup W$ is a subspace $\Leftrightarrow U\subseteq W$ or $W\subseteq U$.

$\Leftarrow$) If $U\subseteq W\Rightarrow U\cup W=W$ and this is a subspace

If $U\supseteq W\Rightarrow U\cup W=U$ and is a subspace.

$\Rightarrow$). Suppose $U\cup W$ is a subspace. Suppose by contradiction that $U\not\subseteq W$ and $W\not\subseteq U.\Rightarrow W\setminus U\neq\emptyset$. $U\setminus W\neq\emptyset$.

$\Rightarrow\exists w\in W\setminus U$ and $\exists u\in U\setminus W$. In particular, u and $w\in W\cup U\Rightarrow u+w\in U\cup W$.

If $u+w\in U$, since $u\in U,\Rightarrow u+w+(-u)=u+(-u)+w=w\in U$. Contradiction.

If $u+w\in W$ since $w\in W,\Rightarrow u+w+(-w)=u\in W$ Contradiction.

Therefore $U\subseteq W$ or $W\subseteq U$.

3.27.

Definition: Suppose $U_1 \dots U_m$ subspace of V. we denote $U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_i \in U_i\}$ (which is also a subspace of V.)

① Describe the sum $U+W$ in the following cases:

a) $U = \{(x,0,0) \in \mathbb{R}^3 : x \in \mathbb{R}\}$. $W = \{(y,y,0) \in \mathbb{R}^3 : y \in \mathbb{R}\}$.

An element $v \in U+W$ is of the form:

$v = u + w$ $\quad u = (x,0,0) \in U$ $\quad w \in (y,y,0) \in W$.

$\Rightarrow v = (x+y, y, 0)$.

Now we can write $v = \{\bar{x}, y, 0\}$ for $\bar{x}, y \in \mathbb{R}$. $\Rightarrow U+W = \{(\bar{x}, y, 0) \in \mathbb{R}^3, x, y \in \mathbb{R}\}$.

b). $U = \{(x,y,z) \in \mathbb{R}^3 : x-y+z=0 \wedge 2y+z=0\}$. $W = \{(x,y,z) \in \mathbb{R}^3 : x-3y+1z=0 \wedge 2x-3y+z=0 \wedge -y+3z=0\}$.

$\cdot (x,y,z) \in U \Leftrightarrow \begin{cases} x-y+z=0 \\ 2y+z=0 \end{cases} \Leftrightarrow \begin{cases} x=3y \\ z=-2y \end{cases} \Rightarrow$ So $U = \{(3y, y, -2y) \in \mathbb{R}^3 : y \in \mathbb{R}\}$. In the same way

we have $(x,y,z) \in W \Leftrightarrow \begin{cases} x=4z \\ y=3z \end{cases} \Rightarrow W = \{(4z, 3z, z) \in \mathbb{R}^3 : z \in \mathbb{R}\}$.

Therefore every $v \in U+W$ is of the form: $v = (3y, y, -2y) + (4z, 3z, z) = y(3,1,-2) + z(4,3,1)$.

these vectors are not multiplies of each other.

$\Rightarrow U+W = \{y(3,1,-2) + z(4,3,1), y, z \in \mathbb{R}\}$.

Note: If U_1 is a subspace of $U_2 \Rightarrow U_1 + U_2 = U_2$.

$\subseteq$) let $x \in U_1 + U_2 \Rightarrow x = u_1 + u_2$ for $u_1 \in U_1$ $u_2 \in U_2 \Rightarrow u_1 \in U_2 \Rightarrow x \in u_1 + u_2 \in U_2$ So $U_1 + U_2 \subseteq U_2$.

$\supseteq$) let $u_2 \in U_2 \Rightarrow u_2 = \underset{\in U_1}{0} + \underset{\in U_2}{u_2} \in U_1 + U_2$. So $U_2 \subseteq U_1 + U_2$

$\Rightarrow U_2 = U_1 + U_2$.

Definition:

let $U_1 \dots U_m$ subspace of V, we say that the sum $U_1 + U_2 + \dots + U_m$ is a direct sum if $x \in U_1 + U_2 + \dots + U_m$ can be only one way as $x = u_1 + u_2 + \dots + u_m$ for $u_i \in U_i$. We denote as $U_1 \oplus U_2 \oplus U_3 \oplus \dots U_m$.

Theorem: if U, W are subspaces of V then $V = U \oplus W \Leftrightarrow V = U+W$ and $U \cap W = \{0\}$.

② If $V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is a function}\}$. and $U = \{f \in V, f \text{ is even}\}$. $W = \{f \in V, f \text{ is odd}\}$.

Prove that $V = U \oplus W$.

- $V = U + W$. let $f \in V$. we want $f(x) = g(x) + h(x)$ for $g \in U$, $h \in W$.

$f(x) = f(x) + f(-x) - f(-x) = \frac{f(x)}{2} + \frac{f(-x)}{2} + \frac{f(x)}{2} - \frac{f(-x)}{2} = \underbrace{\tfrac{1}{2}(f(x) + f(-x))}_{g(x)} + \underbrace{\tfrac{1}{2}(f(x) - f(-x))}_{h(x)}$

So we can take:

$g(x) = \frac{1}{2}(f(x) + f(-x))$. is even. $h(x) = \frac{1}{2}(f(x) - f(-x))$ is odd. $g \in U$. $h \in W$. $\Rightarrow f = g + h \Rightarrow V = U + W$.

- $U \cap W = \{0\}$. let $g \in U \cap W \Rightarrow g \in U$ and $g \in W$.

$\Rightarrow g(x) = g(-x)$. $\wedge$ $g(x) = -g(-x)$. $\Rightarrow g(-x) = -g(-x) \Rightarrow g(x) = -g(x)$. $\forall x \in \mathbb{R} \Rightarrow g(x) = 0$ $\forall x \in \mathbb{R}$

$\Rightarrow g$ is a 0 function.

Therefore $V = U \oplus W$.

④ Suppose $U = \{(x,x,z,z) \in \mathbb{R}^4 : x, z \in F\}$. Find a subspace W of F^4 st $F^4 = U \oplus W$.

We want to write any vector $(x,y,z,w) \in F^4$ in the form $(x,y,z,w) = u + w$ for $u \in U$, $w \in W$.

Find every $u \in U \Rightarrow u = (x,x,y,y)$ and so the 2nd and the 4th coordinates are determined by 1st and 3rd coordinates.

So we can find any $(\bar{x}, y, \bar{z}, w) \in F^4$ with U. so for W we should add the 2nd and 4th coordinate So we define $W = \{(0,y,0,w) : y, w \in F\}$.

let's see that $F^4 = U \oplus W$

$\cdot F^4 = U + W$ for any $(x,y,z,w) \in F^4$. we can write: $(\bar{x}, y, \bar{z}, w) = (x,x,z,z) + (0, y-x, z, w-z)$. $\in U+W$.

$\cdot U \cap W = \{0\}$. let $(x,y,z,w) \in U \cap W \Rightarrow (x,y,z,w) \in U \wedge (x,y,z,w) \in W \Rightarrow x=y$, $z=w$. $\wedge$ $y=0$, $z=0 \Rightarrow (x,y,z,w) = 0$. ☒

Definition: let $v_1 v_2 \cdots v_n \in V$ vectors, the SPAN of $v_1, v_2 \cdots v_n$ is the smallest subspace contain $v_1, v_2 \cdots v_n$. We denote it as $\langle v_1, v_2, \ldots v_n \rangle = \text{span}\{v_1 v_2 \ldots v_n\}$.

Definition: let $v_1 v_2 \cdots v_n \in V$. $w \in V$ is a linear combination of $v_1 v_2 \cdots v_n$ if $w = \lambda_1 v_1 + \cdots + \lambda_n v_n$ for $\lambda_i \in F$.

Observation: $\langle v_1 \ldots v_n \rangle$ is the set of the linear combination.

That is $w \in \langle v_1, v_2 \cdots v_n \rangle \Leftrightarrow w = \lambda_1 v_1 + \cdots + \lambda_n v_n$ for $\lambda_i \in F$.

④ Is the vector $w = (0, -1, 3) \in \mathbb{R}^3$ in the subspace spanned $\langle v_1 v_2 \rangle$ by $v_1 = (1, -3, 5)$ $v_2 = (2, -3, 1)$?

We know that $w \in \langle v_1 v_2 \rangle \Leftrightarrow w = \lambda_1 v_1 + \lambda_2 v_2$ for $\lambda_i \in \mathbb{R}$. $\Leftrightarrow (0, -1, 3) = \lambda_1 (1, -3, 5) + \lambda_2 (2, -3, 1)$. $\Rightarrow \begin{cases} 0 = \lambda_1 + 2\lambda_2 \\ -1 = -3\lambda_1 - 3\lambda_2 \\ 3 = 5\lambda_1 + \lambda_2 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = -2\lambda_2 \\ -1 = 3\lambda_2 \\ 3 = -9\lambda_2 \end{cases} \Leftrightarrow \begin{matrix} \lambda_1 = \frac{2}{3} \\ \lambda_2 = \frac{-1}{3} \end{matrix}$

Therefore $w = \frac{2}{3} v_1 - \frac{1}{3} v_2$ and so $w \in \langle v_1, v_2 \rangle$.

⑤ Consider $v = (1, 2, 3)$. $u = (2, 3, 1) \in \mathbb{R}^3$. Is it possible to find conditions on $a, b, c \in \mathbb{R}$ st $(a, b, c) = w \in \mathbb{R}^3$ is in the subspace spanned by v and u?

We have that $w \in \langle v, u \rangle \Leftrightarrow w = \lambda_1 v + \lambda_2 u$. $\Leftrightarrow (a, b, c) = \lambda_1 (1, 2, 3) + \lambda_2 (2, 3, 1)$. $\Leftrightarrow \begin{cases} a = \lambda_1 + 2\lambda_2 \\ b = 2\lambda_1 + 3\lambda_2 \\ c = 3\lambda_1 + \lambda_2 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = a - 2\lambda_2 \\ \lambda_2 = 2a - b \\ c = -7a + 5b \end{cases}$

So $w = (a, b, c) \in \langle v, u \rangle \Leftrightarrow c = -7a + 5b$. In this case, $w = (-3a + 2b) \cdot v + (2a - b) u$.

Remainder: $\mathbb{C}^2$ is a vector space over $F = \mathbb{C}$ but $\mathbb{C}^3$ also a vector space over $F = \mathbb{R}$.

⑥ let $v = (1, 1 + i)$ $u = (i, -i)$ $w = (0, 1 - 2i) \in \mathbb{C}^2$.

a) Consider $\mathbb{C}^2$ as a real vector space $(F = \mathbb{R})$ is $v = \langle u, w \rangle$?

$v \in \langle u, w \rangle \Rightarrow v = \lambda \cdot u + \alpha \cdot w$ for $\lambda, \alpha \in \mathbb{R}$. $\Leftrightarrow \begin{cases} 1 = \lambda \cdot i + \alpha \cdot 0 \\ 1 + i = \lambda i + \alpha \cdot (1 - 2i) \end{cases}$ and there is not possible since $\lambda i \notin \mathbb{R}$ (if $\lambda \in \mathbb{R}$).

So $v \notin \langle u, w \rangle$ for $\mathbb{C}^2$ as a real vector space.

b) Considering $\mathbb{C}^2$ as a complex vector space is $v \in \langle u, w \rangle$?

$v \in \langle u, w \rangle \Leftrightarrow v = \lambda \cdot u + \alpha \cdot w$ for $\lambda, \alpha \in \mathbb{C}$. $\Leftrightarrow \begin{cases} 1 = \lambda \cdot i \\ 1 + i = -\lambda i + \alpha (1 - 2i) \end{cases} \Leftrightarrow \begin{cases} \lambda = -i \\ \alpha = i \end{cases}$

⑦ Let $\{v_1, v_2 \ldots v_n\} \subseteq V$ and $\{w_1 w_2 \ldots w_k\} \in V$ Prove that if $\{v_1, v_2 \cdots v_n\} \subseteq \{w_1 w_2 \cdots w_k\} \Rightarrow \langle v_1 \ldots v_n \rangle \subseteq \langle w_1 \ldots w_k \rangle$

let $v \in \langle v_1 v_2 \cdots v_n \rangle$. $\Rightarrow \exists \lambda_1 \ldots \lambda_n \in F$. st $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n \cdot v_n$.

Since $\{v_1, \ldots, v_n\} \subseteq \{w_1 w_2 \ldots w_k\} \Rightarrow \{v_1 v_2 \cdots v_n, v_{n+1} \ldots v_k\} = \{w_1 w_2 \cdots w_k\}$ and so we can also considers:

$v = \lambda_1 v_1 + \cdots + \lambda_n v_n + 0 \cdot v_{n+1} + 0 \cdot v_{n+2} + \cdots + 0 \cdot v_k$. $\Rightarrow$ linear combination of $\langle v_1 v_2 \cdots v_k \rangle = \{w_1 w_2 \cdots w_k\}$. $\Rightarrow v \in \langle w_1, w_2 \cdots w_k \rangle$

$\boxtimes$

8. let $U = \{u_1 u_2 \cdots u_n\} \subseteq V$, $W = \{w_1 w_2 \cdots w_k\} \subseteq V$. Prove that $\langle U \cup W \rangle = \langle U \rangle + \langle W \rangle$

$\subseteq$). let $v \in \langle U \cup W \rangle \Rightarrow \exists \lambda_1 \lambda_2 \cdots \lambda_m \in F$ st $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m$ for $v_i \in U \cup W$. Each $v_i \in U \cup W \Rightarrow v_i \in U$ or $v_i \in W$.

So we can reorder the set $\{v_1 v_2 \ldots v_m\}$ and assume without loss of generality. $v_1 \cdots v_t \in U$ and $v_{t+1} \ldots v_m \in W$.

So $v = \underbrace{\lambda_1 v_1 + \cdots + \lambda_t v_t}_{\in \langle U \rangle} + \underbrace{\lambda_{t+1} v_{t+1} + \cdots + \lambda_m v_m}_{\in \langle W \rangle}$.

3.28.1

Definition:

A set of vector $\{v_1 \ldots v_m\} \subseteq V$ is called linearly independent (LI) if $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = 0$ for $\lambda_i \in F$ this implies $\lambda_1 = \lambda_2 = \cdots \lambda_n = 0$.

A set of vectors $\{v_1 v_2 \cdots v_n\} \subseteq V$ is called linearly dependent (LD) if it is not LD. That is $\exists \lambda_1 \lambda_2 \ldots \lambda_n \in F$. not all zero st $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$.

① Check that the following vectors are LI or not. If not, find a non-trivial linear combination equal to 0.

a) $v_1 = (2, 2, 2)$. $v_2 = (0, 0, 3)$ $v_3 = (0, 1, 1)$ in $\mathbb{R}^3$.

Suppose $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$. $\Rightarrow \lambda_1 (2, 2, 2) + \lambda_2 (0, 0, 3) + \lambda_3 (0, 1, 1) = (2\lambda_1, 2\lambda_1 + \lambda_3, 2\lambda_1 + 3\lambda_2 + \lambda_3) = (0, 0, 0)$.

$\Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ \lambda_3 = 0 \end{cases} \Rightarrow \{v_1, v_2, v_3\}$ is LI.

b) $p_1(x) = 2-x^2$, $p_2(x) = 3x$, $p_3(x) = x^2+x-2$. in $\mathbb{R}[x]$.

Suppose $\lambda_1 p_1(x) + \lambda_2 p_2(x) + \lambda_3 p_3(x) = 0$.

$\lambda_1(2-x^2) + \lambda_2 \cdot 3x + \lambda_3(x^2+x-2) = 0 \iff (\lambda_3-\lambda_1)x^2 + (3\lambda_2+\lambda_3)x + 2\lambda_1 - 2\lambda_3 = 0$.

$\iff \lambda_3 = \lambda_1$. $3\lambda_2 = -\lambda_3$. $2\lambda_1 = 2\lambda_3$. $\iff \lambda_1 = \lambda_3$, $\lambda_2 = -\frac{1}{3}\lambda_3$. So we don't get the only λ.

Whenever we have $\lambda_1 = -3\lambda_2 = \lambda_3$ for $\lambda_2 \in \mathbb{R}$. We get $\lambda_1 p_1(x) + \lambda_2 p_2(x) + \lambda_3 p_3(x) = 0$.

So a non-trivial linear combination would be taken as $\lambda_2 = 1$, $\lambda_1 = \lambda_3 = 3$.

c) $\cos$, $\sin$, id in $V = \{f: \mathbb{R} \to \mathbb{R}$. f function$\}$.

Suppose $\lambda_1 \cos + \lambda_2 \sin + \lambda_3 id = 0$ function.

$\Rightarrow \lambda_1 \cos x + \lambda_2 \sin x + \lambda_3 x = 0$. $\forall x \in \mathbb{R}$. We can evaluate in specific value of x.

- $x=0$ $\lambda_1 \cos(0) + \lambda_2 \sin(0) + \lambda_3 \cdot 0 = 0$. $\Rightarrow \lambda_1 = 0$.
- $x = \pi$. $\lambda_1 \cos\pi + \lambda_2 \sin(\pi) + \lambda_3 \pi = 0 \Rightarrow \lambda_3 \pi = 0$ since $\lambda_1 = 0$, $\lambda_2 \sin\pi = 0$.
- $x = \frac{\pi}{2}$. $\lambda_1 \cos\frac{\pi}{2} + \lambda_2 \sin(\frac{\pi}{2}) + \lambda_3 \cdot x = 0 \Rightarrow \lambda_2 = 0$.

Therefore $\{\cos, \sin, id\}$ are LI.

② Prove the following:

a) Every subset of a LI set is also LI.

let $\{v_1 \dots v_n\}$ is a LI. and $\{w_1, w_2, w_3 \cdots w_k\} \subseteq \{v_1, v_2 \cdots v_n\}$. a subset suppose $\tilde{\lambda}_1 w_1 + \tilde{\lambda}_2 w_2 + \dots + \tilde{\lambda}_k w_k = 0$

We can consider $\{w_1, \cdots w_k, w_{k+1}, \cdots w_n\} = \{v_1, v_2 \cdots v_n\}$.

So now $0 = \lambda_1 w_1 + \lambda_2 w_2 + \cdots + 0 \cdot w_{k+1} + 0 \cdot w_{k+2} + \cdots + 0 \cdot w_n$. Since $\{v_1, v_2 \dots v_n\}$ is LI and we have a linear combination of there equal to 0. We have that $\lambda_1 = \lambda_2 = \cdots + \lambda_n = 0$.

Since we started with $\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_k w_k = 0$. $\Rightarrow \{w_1, w_2 \cdots w_k\}$ are LI by definition.

b) Every set that contains a LD subset is also LD.

Suppose that we have $\{v_1, v_2 \cdots v_n\}$ a set and that the subset $\{v_1, v_2 \cdots v_k\}$ is LD. ($\{v_1, v_2 \cdots v_k\} \subseteq \{v_1 \dots v_n\}$).

Since $\{v_1, v_2 \dots v_n\}$ is LD. $\Rightarrow \exists \lambda_1, \lambda_2 \cdots \lambda_k \in F$ not all zero st $\lambda_1 v_1 + \lambda_2 v + \cdots + \lambda_k v_k = 0$. $\Rightarrow \tilde{\lambda}_1 v_1 + \tilde{\lambda}_2 + \cdots + \tilde{\lambda}_k v_k + \cdots 0 v_n = 0$

$\Rightarrow \{v_1 \dots v_n\}$ is LD.

c) Every set containing 0 is LD.

Suppose we have $\{0, v_1, v_2 \dots v_n\}$ we can contain $1 \in F$ (or any element in F) st: $0 = 1 \cdot 0 + 0 \cdot v_1 + 0 \cdot v_2 + \dots 0 v_n$

$\Rightarrow$ the scalars are $1, 0, 0, 0 \cdots 0$ are not all zero $\Rightarrow$ LD.

③ Suppose that $\{v_1, v_2 \cdots v_n\}$ are LI. and for $w \in V$. $\{v_1+w, v_2+w, \cdots v_n+w\}$ is LD. Prove that $w \in \langle v_1, v_2 \dots v_n \rangle$.

Since $\{v_1+w, v_2+w, \cdots v_n+w\}$ is L.D. $\Rightarrow \exists \lambda_1 \dots \lambda_n \in F$ not all zero st $\lambda_1(v_1+w) + \lambda_2(v_2+w) + \dots + \lambda_n(v_n+w) = 0$.

$\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n + \lambda_1 w + \lambda_2 w + \cdots + \lambda_n w = 0$. $\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = -(\lambda_1 + \lambda_2 + \cdots + \lambda_n) w$

Suppose we have $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$. $\Rightarrow \lambda_1 v_1 + \cdots + \lambda_n v_n = 0$. with $\lambda_1 \cdots \lambda_n$ all not zero. $\Rightarrow \{v_1, v_2, \cdots v_n\}$ are LD. Contradiction

So $\lambda_1 + \lambda_2 + \cdots + \lambda_n \neq 0$. $\Rightarrow \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = -(\lambda_1 + \lambda_2 + \cdots \lambda_n) w$. $\Rightarrow \frac{-\lambda_1}{\lambda_1 + \cdots \lambda_n} v_1 + \frac{-\lambda_2}{\lambda_1 + \cdots + \lambda_n} v_2 + \cdots + \frac{-\lambda_n}{\lambda_1 + \cdots + \lambda_n} v_n = w$. $\Rightarrow w \in \langle v_1 \dots v_n \rangle$.

4.8.

② Suppose V is finite-dimentional and $U \subseteq V$ is a subspace We know that $\dim(U) \leq \dim(V)$

Prove that if $\dim(U) = \dim(V) \Rightarrow U = V$.

Suppose $\{u_1, \dots u_n\}$ is a basis of U ($\dim(U) = n$). Since $\{u_1, \dots u_n\}$ is LI in U so it is LI in V.

and it has n elements $\Rightarrow \{u_1, u_2 \dots u_n\}$ is a basis in V. $\Rightarrow$ every $v \in V$ we can write as $v = \lambda_1 u_1 + \cdots + \lambda_n u_n$

$\in U$ So $V \subseteq U \subseteq V \Rightarrow U = V$.

③ let $U_1, U_2 \cdots U_m$ finite dimentional subspaces of V. Prove the following:

a) $U_1, U_2 \dots U_m$ finite-dimentional and $\dim(U_1 + U_2 + \dots + U_m) \leq \dim(U_1) + \dim(U_2) + \dots + \dim(U_m) < \infty$.

let $B_k = \{v_1^k, v_2^k, \cdots v_k^k\}$ is a basis of U_k. and consider $B = B_1 \cup \cdots \cup B_m$, let's show that B spans $U_1 + \cdots + U_m$.

let $u_1 + \cdots + u_m \in U_1 + \cdots + U_m$, since every $u_k \in U_k$. and B_k is a basis of U_k we can write

$U_1+\cdots+U_m = \underbrace{(\lambda_1^1 v_1^1+\cdots+\lambda_n^1 v_n^1)}_{B_1} + \cdots + \underbrace{(\lambda_1^m v_1^m+\cdots+\lambda_m^m v_m^m)}_{B_m}$. So B span $U_1+\cdots+U_m$. Since $B=B_1\cup B_2\cup\cdots\cup B_m$.

$B_1 + B_2 + \cdots + B_m$. $B_1 \subseteq B_1\cup B_2\cup\cdots\cup B_m$, $\cdots$ $B_m\subseteq B_1\cup B_2\cup\cdots\cup B_m$.

By lecture, spanning set contains a basis $\Rightarrow \exists \tilde{B}$ basis of $U_1+\cdots+U_m$ st $\tilde{B}\subseteq B = B_1\cup\cdots\cup B_m$.

$\dim(U_1+\cdots+U_m) \leq \dim(U_1)+\dim(U_2)+\cdots+\dim(U_m)$

b) if $U_1+\cdots+U_m$ is a direct sum $\Rightarrow \dim(U_1\oplus U_2\oplus\cdots\oplus U_m) = \dim(U_1)+\dim(U_2)+\cdots+\dim(U_m)$.

let's show that $B=B_1\cup B_2\cup\cdots\cup B_m$ is also LI. Suppose that $(\lambda_1^1 v_1^1+\lambda_2^1 v_2^1+\cdots+\lambda_n^1 v_n^1)+\cdots+(\lambda_1^m v_1^m+\cdots+\lambda_m^m v_m^m)=0$. ①

Since $U_1\oplus U_2\oplus\cdots\oplus U_m$ is a direct sum, every $v\in U_1+U_2+\cdots+U_m$ be written in a unique way. In particular, the only way to write $0\in U_1+U_2+\cdots+U_m$ is $0=0+0+\cdots+0$. So by ①. $\lambda_1^1 v_1^1+\cdots+\lambda_n^1 v_n^1=0$. .... $\lambda_1^m v_1^m+\cdots+\lambda_m^m v_m^m=0$.

Since B_k is LI. So $\lambda_1^1=\lambda_2^1=\cdots=\lambda_1^m=\cdots=\lambda_m^m=0$. Therefore B is LI and by (a) B spans $U_1+U_2+\cdots+U_m \Rightarrow B$ is a basis of $U_1\oplus U_2\oplus\cdots\oplus U_m$ and B has $\dim(U_1)+\dim(U_2)+\cdots+\dim(U_m)$ elements.

$\Rightarrow \dim(U_1\oplus U_2\oplus\cdots U_m) = \dim(U_1)+\cdots+\dim(U_m)$

Note. with (a) (b) we get that $U_1+\cdots+U_m$ is a direct sum $\Leftrightarrow \dim(U_1+U_2\cdots+U_m)=\dim(U_1)+\cdots+\dim(U_m)$.

② Suppose U and W are subspaces of $\mathbb{R}^9$. st $\dim U=\dim W=5$. Prove that $U\cap W\neq\{0\}$.

We know that $\dim(U+W)=\dim U+\dim W-\dim(U\cap W) = 10-\dim(U\cap W)$. Since U and W are subspaces of $\mathbb{R}^9 \Rightarrow \dim(U+W)\leq 9$. $\Rightarrow 10-\dim(U\cap W)\leq 9$. $\Rightarrow \dim(U\cap W)\geq 1$. $\Rightarrow U\cap W\neq\{0\} \Rightarrow$ it has at least one element in it.

③ let U and W subspaces of $\mathbb{C}^6$. We have $\dim U=\dim W=4$. Prove that there exists 2 vectors $v_1,v_2\in U\cap W$ st v_1 is not a scalar multiple of v_2. ($v_1\neq\lambda v_2$)

We have that $\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)$. $= 8-\dim(U\cap W)$. $\Rightarrow \dim(U\cap W)=8-\dim(U+W)$.

Since $U+W$ is a subspace of $\mathbb{C}^6$. $\Rightarrow \dim(U+W)\leq 6$. $\Rightarrow -\dim(U+W)\geq -6$. $\Rightarrow 8-\dim(U\cap W)\geq 2$.

$\Rightarrow \dim(U\cap W)\geq 2$. Therefore $U+W$ has a basis B at least 2 elements $v_1,v_2\in B$. Since B is LI, and every subset of a LI set is also LI. $\Rightarrow \{v_1,v_2\}\subseteq B$. is a LI set $\Rightarrow v_1$ is not a scalar multiple of v_2.

Note: if A,B,C are arbritrary sets and we look at cardinality:

- $|A\cup B| = |A|+|B|-|A\cap B|$.
- $|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$.

Now, as we consider U_1, U_2, U_3 subspaces of V (finite-dimentional) We know that $U_1 \cup U_2$ and $U_1 \cup U_2 \cup U_3$ is not always a subspace.
But U_1+U_2 and $U_1+U_2+U_3$ always be subspace of V.

④ IS it true $\dim(U_1+U_2+U_3) = \dim(U_1)+\dim(U_2)+\dim(U_3) - \dim(U_1\cap U_2) - \dim(U_1\cap U_3) - \dim(U_2\cap U_3) + \dim(U_1\cap U_2\cap U_3)$
Consider $U_1 = \mathbb{R}\times\{0\} = \{(a,0), a\in\mathbb{R}\}$ $U_2 = \{0\}\times\mathbb{R} = \{(0,a)\ a\in\mathbb{R}\}$ and $U_3 = \langle(1,1)\rangle$ subspace of $\mathbb{R}^2$.
Therefore $U_1+U_2+U_3 = \mathbb{R}^2$. $U_1\cap U_2 = \{(0,0)\} = U_1\cap U_3 = U_2\cap U_3 = U_1\cap U_2\cap U_3$. Also $\dim U_1 = \dim U_2 = \dim U_3 = 1$.
$\dim(U_1+U_2+U_3) = \dim(\mathbb{R}^2) = 2$. $\dim(U_1)+\dim(U_2)+\dim(U_3) - \dim(U_1\cap U_2) - \dim(U_1\cap U_3) - \dim(U_2\cap U_3) + \dim(U_1\cap U_2\cap U_3)$
$= 1+1+1-0-0-0+0 = 3$. NOT EQUAL !

Definition: The elementary matrices are matrices obtained by identity and apply one of the following elementary operations.

i) P_{ij}: interchanging row i and row j. $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$.

ii) $M_i(\lambda)$ multiple row i / column i by λ: $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \xrightarrow{2R_1} \begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

iii) $T_{ij}(\lambda)$ replace row R_i by $R_i+\lambda R_j$ $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \longrightarrow \begin{pmatrix}1&0&0\\\lambda&1&0\\0&0&1\end{pmatrix}$. $R_2 = R_2+\lambda R_1$.

⑤ Consider $A = \begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} \in M_3(\mathbb{R})$. Find elementary matrices E_i st:

a) $E_1A = \begin{pmatrix}2&4&6\\4&5&6\\7&8&9\end{pmatrix}$ $E_1 = \begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

b) $E_1A = \begin{pmatrix}1&2&3\\7&11&15\\7&8&9\end{pmatrix}$ $R_2 = R_2+3R_1 \Rightarrow E_1 = \begin{pmatrix}1&0&0\\3&1&0\\0&0&1\end{pmatrix}$

c). $E_2E_1A = \begin{pmatrix}5&7&9\\8&10&12\\7&8&9\end{pmatrix}$. ① $R_1 = R_1+R_2$: $\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix} = E_1$. ② $R_2 = 2\cdot R_2$. $\begin{pmatrix}1&0&0\\0&2&0\\0&0&1\end{pmatrix} = E_2$.

This is not the same as doing first E_2 and E_1!

Definition: Two matrices $A.B$ are row-equivalent if $\exists E_1\ldots E_k$ elementary matrices st $A = E_k\cdots E_1\cdot B$.

⑥ Prove the row-equivalence is equivalent relation.
Consider $A\sim B \Leftrightarrow A$ is row-equivalent to B. $\Leftrightarrow B = E_1\cdots E_kA$. for $E_1\cdots E_k$ elementary matrices.
Reflexive: ARA since $A = Id\cdot A$.
Symmetric: $ARB \Rightarrow BRA$, Since ARB: $B = E_1\cdots E_kA$. Remember that elementary matrices are invertible and their inverses are also elementary matrices. $\cdot P_{ij}^{-1} = P_{ij}$. $\cdot M_i(\lambda)^{-1} = M_i(\frac{1}{\lambda})$.
$T_{ij}(\lambda)^{-1} = T_{ij}(-\lambda)$. So $A = E_1^{-1}\cdots E_k^{-1}\cdot B. \Rightarrow BRA$.
Transitive: $ARB \wedge BRC \Rightarrow B = E_1\cdots E_k\cdot A \wedge C = E_1'\cdots E_k'\cdot B. \Rightarrow C = E_1'\cdots E_k'\cdot E_1\cdots E_k\cdot A \Rightarrow ARC$.

Note. if $ARId \Rightarrow Id = \overbrace{(E_k\cdots E_1)}^{A^{-1}}A. \Rightarrow A$ is invertible.
Therefore the equivalence class of Id is $[Id] = \{A : ARId\} = \{A : A \text{ is invertible}\}$. So to prove A invertible, you find elementary matrices $E_1\cdots E_k$. st $Id = E_1\cdots E_k\cdot A$.

⑦ Determaine the following matrices have inverse.

a) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -3 & -2 & 1 \end{array}\right) \xrightarrow{R_2/-3} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right)$$

So $Id = E_3 \cdot E_2 \cdot E_1 \cdot A$ $\quad A^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$.

b) $A = \begin{pmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 4 & -1 & 2 & 0 & 1 & 0 \\ 6 & 4 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 0 & -11 & 4 & -2 & 1 & 0 \\ 0 & -11 & 4 & -3 & 0 & 1 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|ccc} 2 & 5 & -1 & 1 & 0 & 0 \\ 0 & -11 & 4 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right)$$

$\Rightarrow$ No way to get identity by applying elementary matrices

Therefore, not inverse.

c) $A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 6 \\ -1 & 0 & -10 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 6 & 0 & 1 & 0 \\ -1 & 0 & -10 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2=R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ -1 & 0 & -10 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3=-\frac{1}{6}R_3]{R_3=R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 0 & -2 \\ 0 & 1 & 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & 3/2 & 0 & 1/2 \end{array}\right).$$

① let $A \in M_n(F)$ and suppose A is row-equivalent to B, which has a row of 0's. Prove that A is not inversible.

Since B has a row of 0's. $\Rightarrow B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ 0 & 0 \cdots & 0 \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}$ $\Rightarrow B \cdot C$ also has a row of 0's for any $C \in M_n(F)$.

Since $B \cdot C = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ 0 & 0 \cdots & 0 \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} \cdot \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \\ c_{n1} & \cdots & c_{nm} \end{pmatrix} = \begin{pmatrix} \cdot & - & - \\ 0 & \cdots & 0 \\ - & - & - \end{pmatrix} \to$ row i. $\quad (BC)_{ij} = \sum_{k=1}^{m} b_{ik} \cdot c_{kj} = 0$. Therefore B is not inversible since $B \cdot C \neq Id$. $\forall C \in M_n(F)$. Now A is row-equivalent to $B \Rightarrow A R B \Rightarrow B = E_k \cdots E_1 A \Rightarrow A = E_k^{-1} \cdots E_1^{-1} B$.

Therefore A is not inversible.

Note if B has 0 column $\Rightarrow A$ also not inversible.

② let $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in M_{2\times1}(F)$ and $B = (b_1 \ b_2) \in M_{1\times2}(F)$. Prove that $C = A \cdot B \in M_2(F)$ is not inversible.

We write $C = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (b_1 \ b_2) = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}$. Consider following cases:

i) a_1, a_2, b_1, b_2 is 0. C is not inversible.

ii) $a_1 \neq 0$, $a_2 \neq 0$, $b_1 \neq 0$, $b_2 \neq 0$. $C = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix} \xrightarrow{R_2 - \frac{a_2}{a_1} R_1} \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ 0 & 0 \end{pmatrix} \Rightarrow C$ is row equivalent to a matrix with 0's.

$\Rightarrow C$ is not inversible.

③ let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$. Prove every elementary row operations that A is inversible $\Leftrightarrow ad - bc \neq 0$. In this case,

$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$\Rightarrow$ Suppose that A is inversible and by contradiction $ad - bc = 0$. Since $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is inversible $\Rightarrow a \neq 0$ or $c \neq 0$. (if $a = c = 0$. A has a 0 column $\Rightarrow A$ is not inversible), we can suppose that $a \neq 0$.

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 - \frac{c}{a} R_1} \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix} \xrightarrow{R_2 \cdot a} \begin{pmatrix} a & b \\ 0 & ad-bc \end{pmatrix}$. Since $ad - bc = 0$. $\Rightarrow C = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \Rightarrow$ irvinversible $\Rightarrow$ Contradiction.

So $ad - bc \neq 0$.

$\Leftarrow$). Since $ad - bc \neq 0$. $\Rightarrow a \neq 0$ or $c \neq 0$. Suppose without loss of generality that $a \neq 0$ let's find the inverse of A.

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow[R_2 - \frac{c}{a} R_1]{} \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix} \xrightarrow{R_2 \cdot a} \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right) \xrightarrow{R_2 \cdot \frac{1}{ad-bc}} \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \xrightarrow{R_1 - b \cdot R_2} \left(\begin{array}{cc|cc} a & 0 & 1 + \frac{cb}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \xrightarrow{R_1 \cdot \frac{1}{a}} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$\Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$

* Determaine for which value of the following matrices are inversible.

a) $A = \begin{pmatrix} a & a \\ a & 1 \end{pmatrix}$

iff $a \cdot 1 - aa \neq 0. \Rightarrow a(1-a) \neq 0. \Leftrightarrow a \neq 0$ and $a \neq 1$.

b) $A = \begin{pmatrix} a^2 & 1 \\ 2a & a \end{pmatrix}$

$\Leftrightarrow a^2 \cdot a - 1 \cdot 2a \neq 0 \Leftrightarrow a(a^2-2) \neq 0. \Rightarrow a \neq 0$ and $a \neq \sqrt{2}$.

4.24.

Given a system of linear equivalians.

$$\begin{cases} A_{11}x_1 + \cdots + A_{1n}x_n = b_1 \\ \vdots \\ A_{m1}x_1 + \cdots + A_{mn}x_n = b_n \end{cases}$$

$\Rightarrow$ we can consider the equivalent matrix form: $Ax = b$.

$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \in M_{m\times n}(F)$. $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n$. $\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in F^m$.

Now apply 3 elementary row operations to A and b. which don't change the space of solution

1) Interchange rows.
2) Scalar multiplication.
3) Row sum.

we get $\hat{A} \cdot x = \hat{b}$

GOAL: Get the $\hat{A}$ in the row-reduced echelon form.

① For the following systems of linear equatians write into matrix form and solve.

a) $\begin{cases} x + 2y = 0. \\ 2x + 3y = 0. \end{cases} \Rightarrow \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$ ①

Now apply row operations:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \xrightarrow{R_2 \cdot (-1)} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 2R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So ① is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only solution.

b) $\begin{cases} x - 3y + 5z = 0 \\ 2x - 3y + z = 0 \\ \quad\; y + 3z = 0. \end{cases}$

We have the matrix form: $\begin{pmatrix} 1 & -3 & 5 \\ 2 & -3 & 1 \\ 0 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

$$\begin{pmatrix} 1 & -3 & 5 \\ 2 & -3 & 1 \\ 0 & -1 & 3 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -3 & 5 \\ 0 & 3 & -9 \\ 0 & -1 & 3 \end{pmatrix} \xrightarrow{R_2/3} \begin{pmatrix} 1 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 + 3R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

So we have $\begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = 4z. \; y = 3z.$ So my solutions are $(4z, 3z, z)$

$= z(4, 3, 1)$. for $z \in \mathbb{R}$. or $\langle (4,3,1) \rangle$.

c) $\begin{cases} 3x-y+2z=1 \\ 2x+y+z=i \\ x-3y=0 \end{cases}$ Now. $\left(\begin{array}{ccc|c} 3 & -1 & 2 & 1 \\ 2 & 1 & 1 & i \\ 1 & -3 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 2 & 1 & 1 & i \\ 3 & -1 & 2 & 1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 7 & 1 & i \\ 0 & 8 & 2 & 1 \end{array}\right)$

$\xrightarrow{R_3-\frac{8}{7}R_2} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 7 & 1 & i \\ 0 & 0 & \frac{6}{7} & 1-\frac{8}{7}i \end{array}\right) \xrightarrow{R_3 \cdot \frac{7}{6}} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 7 & 1 & i \\ 0 & 0 & 1 & \frac{7-8i}{6} \end{array}\right) \xrightarrow{R_2-R_3} \xrightarrow{R_2 \cdot \frac{1}{7}} \xrightarrow{R_1+3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-1+2i}{2} \\ 0 & 1 & 0 & \frac{-1+2i}{6} \\ 0 & 0 & 1 & \frac{7-8i}{6} \end{array}\right)$

So the solution is $\left(\frac{-1+2i}{2}, \frac{-1+2i}{6}, \frac{7-8i}{6}\right)$.

Remainder: Homogenous systems ($Ax=0$) always have at least one solution ($x=0$).
Non-Homogenous may not.

② For each of the following systems describe the set of vectors (b_1, b_2) or (b_1, b_2, b_3) have a solution.

a) $\begin{cases} x+y=b_1 \\ 2x+2y=b_2 \end{cases}$ $\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 2 & b_2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 0 & b_2-2b_1 \end{array}\right)$.

So it has a solution $\Leftrightarrow$ the 2-row "makes sense" $\Leftrightarrow 0=b_2-2b_1$. So the set of vector for which have solution $\{(b_1, 2b_1),\ b\in\mathbb{R}\}$.

b) $\begin{cases} x-y+2z+w=b_1 \\ 2x+2y+z-w=b_2 \\ 3x+y+3z=b_3 \end{cases}$

$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & b_1 \\ 2 & 2 & 1 & -1 & b_2 \\ 3 & 1 & 3 & 0 & b_3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & b_1 \\ 0 & 4 & -3 & -3 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right)$ So solution $= b_3-b_2-b_1=0$.

$\Rightarrow \langle (b_1\ b_2\ b_2+b_1)\rangle$

④ Find the value of $k\in\mathbb{R}$ for which the matrix system $\begin{pmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is:
i) Inconsistant (has no solution). ii) Consistant dependent (has more than one solution).
iii) Consistant independent (has a unique solution).

$\left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & 1 \end{array}\right) \xrightarrow[R_3-kR_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & 1-k \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & -k^2-k+2 & 1-k \end{array}\right)$

$-k^2-k+2 = -(k-1)(k+2)$

Now, we consider cases:

- $k=1 \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ solution is $(x,y,z) = (1-y-z, y, z) = (1,0,0)+y(-1,1,0)+z(-1,0,1)$.

In this case, it is consistant dependent.

- $k=-2. \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right) \Rightarrow 3=0$ Contradiction! No solution $\Rightarrow$ inconsistant.

- $k\neq 1, k\neq -2$: $\left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & -(k-1)(k+2) & 1-k \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 0 & 1 & \frac{1}{k+2} \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & k-1 & 0 & \\ 0 & 0 & 1 & \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/(k+2) \\ 0 & 1 & 0 & 1/(k+2) \\ 0 & 0 & 1 & 1/(k+2) \end{array}\right)$ So it is Consistant dependent.

Find a system of linear equations so the set of all solutions is given by $S=\{(1-t, 2+t, 3+t) : t\in\mathbb{R}\}$.
Every solution is of the form $(x,y,z)=(1-t, 2+t, 3+2t)=(1,2,3)+t(-1,1,2)$.

Remainder: let $Ax=b$ a system of linear equations (in the matrix form) and $S=\{x\in F^n : Ax=b\} \to$ set of solutions of $Ax=b$
$S_0=\{x\in F^n ; Ax=0\} \to$ set of solutions of corresponding homogenous system.

Theorem: $S=\{x_0+x : x\in S_0\}$ where x_0 is a particular solution to $Ax=b$ "$S=x_0+S_0$".

4.25.

Definition: let V, W two vector spaces, a function $T: V \to W$ is called a linear map if it satisfies:

1) $T(v_1+v_2) = T(v_1) + T(v_2)$, $\forall v_1, v_2 \in V$.

2) $T(\lambda v) = \lambda T(v)$ $\forall \lambda \in F$, $v \in V$.

We denote $Hom_F(V, W) = \{T: V \to W,\ T \text{ is linear map}\}$. is a vector space.

① Describe the following functions are linear map.

a) $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(x,y) = (x+1, y)$.

Not a LM. for $(1,0) \in \mathbb{R}^2$. $2 \in \mathbb{R}$. then $T(2\cdot(1,0)) = T(2,0) = (3,0)$.

$2T(1,0) = 2\cdot(2,0) = (4,0)$ not equal.

b). $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(x_1, x_2 \dots x_n) = (x_1, x_1x_2, x_1x_2x_3, \dots, x_1x_2\cdots x_n)$.

$T(1,0,0,\dots 0) = (1,0,0\dots 0)$. $\Rightarrow T(1,0,0,\dots 0) + T(0,1,0 \dots 0) = (1,0,0\dots 0)$.

$T(0,1,0,\dots 0) = (0,0,0\dots 0)$.

but $T((1,0,0\dots 0)+(0,1,0\dots 0)) = T(1,1,0,0\dots 0) = (1,1,0\dots 0)$. not equal.

c) $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (y, x, x-2y)$.

• $T((x,y)+(x',y')) = T(x+x', y+y') = (y+y', x+x', x+x'-2(y+y'))$.

$= (y, x, x-2y) + (y', x', x'-2y') = T(x,y) + T(x',y')$ ✓

• $T(\lambda(x,y)) = T(\lambda x, \lambda y) = (\lambda y, \lambda x, \lambda x - 2(\lambda y)) = \lambda(y, x, x-2y) = \lambda T(x,y)$.

Observation. Any function $T: \mathbb{R}^n \to \mathbb{R}^m$ defind as $T(x_1, x_2 \dots x_n) = \left(\sum_{i=1}^n \lambda_i x_i, \sum_{i=1}^n \alpha_i x_i \dots\right)$,
is a LM.

② let V be a vector space with $\dim(V) = 1$. (for example, $V = \mathbb{R}, \mathbb{Q}, \mathbb{C} \dots$).

If $T: V \to V$ is a LM $\Rightarrow T(v) = \lambda v$. $\forall v \in V$. for some $\lambda \in F$.

Since $\dim = 1$. $\exists B = \{w\}$ a basis with only one element, by definition of a basis, every $v \in V$ can be written as $v = \alpha w$. for some $\alpha \in F$. $\Rightarrow T(v) = T(\alpha w) = \alpha T(w)$.

Note that $T(w) \in V$ $(T: V \to V)$. $\Rightarrow T(w) = \lambda \cdot w$ for some $\lambda \in F$.

So $T(v) = \alpha T(w) = \alpha \cdot \lambda \cdot w = \lambda \cdot (\alpha \cdot w) = \lambda \cdot v$.

$\Rightarrow T(v) = \lambda \cdot v$ $\forall v \in V$.

example. $T: \mathbb{R} \to \mathbb{R}$: $T(x) = \cos x$ is not LM.

$T: \mathbb{C} \to \mathbb{C}$: $T(x) = ix$ is LM.

③ Does there exists a LM that satisfies the given properties?

a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ st $T(1,2) = (3,0)$ and $T(3,6) = (0,1)$.

Not exist. $T(3,6) = 3T(1,2) = 3(3,0) = (9,0) \neq (0,1)$. !

b). $T: \mathbb{R}^2 \to \mathbb{R}^4$ st $T(0,1) = (1,2,0,0)$. and $T(1,0) = (1,1,0,0)$.

Note that $\{(0,1), (1,0)\}$ is a basis of $\mathbb{R}^2$. any $(x,y) \in \mathbb{R}^2$ can be written as $(x,y) = y(0,1) + x(1,0)$.

So if we want $T: \mathbb{R}^2 \to \mathbb{R}^4$ LM. satisfying the properties we have note that:

$T(x,y) = T(y(0,1) + x(1,0)) = yT(0,1) + xT(1,0) = (y+x, 2y+x, 0, 0)$. satisfy the properties.

c). $T: \mathbb{R}^3 \to \mathbb{R}$ st $T(1,-1,1) = 2$. $T(1,1,1) = 1$.

Note that $\{(1,1,1), (1,-1,1)\}$ is LI in $\mathbb{R}^3$. but not a basis ($\dim \mathbb{R}^3 = 3$). So we can add a vector:

$\{(1,1,1), (1,-1,1), (1,0,0)\}$. is a basis of $\mathbb{R}^3$. So any $(x,y,z) \in \mathbb{R}^3$ can be written as

$(x,y,z) = \frac{z-y}{2}(1,-1,1) + \frac{y+z}{2}(1,1,1) + (x-z)(1,0,0)$.

If we want T LM satisfying the properties.

$\Rightarrow T(x,y,z) = \frac{z-y}{2}\cdot 2 + \frac{y+z}{2}\cdot 1 + (x-z)\cdot T(1,0,0)$

And now define $T(1,0,0)$ for any vectiors Then it is LM.