Guangdong Technion
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Algebra A

Spring 2025

Homework 4

1) (30 pts) Let V be a $\mathbb{F}$-vector space such that $\dim(V) = n$ and let $U \subseteq V$ be subspace of V such that $\dim(U) = n - 1$.

(a) Prove that if $v \notin U$, then $V = U \oplus \langle v \rangle$.

(b) Prove that if W is a subspace of V such that $W \not\subseteq U$, then $V = U + W$.

2) (30 pts) Let $U = \{p \in \mathrm{P}_4(\mathbb{F}) \colon p(2) = p(5) = p(6)\}$.

(a) Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathrm{P}_4(\mathbb{F})$.

(c) Find a subspace W of $\mathrm{P}_4(\mathbb{F})$ such that $\mathrm{P}_4(\mathbb{F}) = U \oplus W$.

3) (30 pts) Consider the vector space of 2×2 matrices $\mathrm{M}_2(\mathbb{R})$. Let W_1 be the set of matrices of the form $\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$ and let W_2 be the set of matrices of the form $\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$.

(a) Prove that W_1 and W_2 are subspaces of $\mathrm{M}_2(\mathbb{R})$.

(b) Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$. Give an explicit basis for each of these subspaces.

4) (10 pts) Consider the real vector space $\mathrm{M}_2(\mathbb{F})$. Find a basis $\{A_1, A_2, A_3, A_4\}$ for $\mathrm{M}_2(\mathbb{F})$ such that $A_k^2 = A_k$ for each $k = 1, 2, 3, 4$.