HOMEWORK 3

1) (30 pts) Find a spanning list of vectors for $S+T$ as a subspace of the $\mathbb{F}$-vector space V for the following subspaces S and T.

(a) $S = \{(x,y,z) \in \mathbb{R}^3 \colon 2x - 3y + 2z = 0\}$ and $T = \{(x,y,z) \in \mathbb{R}^3 \colon x + z = 0\}$ where $V = \mathbb{R}^3$, $\mathbb{F} = \mathbb{R}$.

(b) $S = \langle (2,-i), (1,i), (1+i,-i) \rangle$ and $T = \langle (i,0), (i,2i), (0,1) \rangle$ where $V = \mathbb{C}^2$, $\mathbb{F} = \mathbb{R}$.

2) (30 pts) Consider the following subspace of $\mathbb{F}^5$:

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbb{F}^5 \colon x, y \in \mathbb{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbb{F}^5$ (none of them equal to $\{0\}$) such that $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

3) (25 pts) Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{F})$ be an upper triangular matrix, that is $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$.
Suppose that the diagonal entries of A are not equal to 0 ($a_{ii} \neq 0$ for every $i = 1, \dots, n$) and denote by $v_1, \dots, v_n$ the columns of A. Prove that $\{v_1, \dots, v_n\}$ is linearly independent and spans every vector in $\mathbb{F}^n$.

4) (15 pts) Prove or give a counterexample: if $\{v_1, v_2, \dots, v_n\}$ is a linearly independent list of vectors in V, then

$$\{5v_1 - 4v_2, v_2, v_3, \dots, v_n\}$$

is linearly independent.