HOMEWORK 2

1) (45 pts) Check whether the following sets V with the defined operations are vector spaces over $\mathbb{F}$ or not. If they are vector spaces, prove the properties. If not, what property fails?

(a) $V = \mathbb{R}_{>0}$, $\mathbb{F} = \mathbb{Q}$ with

$$a \oplus b = a.b$$
$$\frac{n}{m} \odot a = \sqrt[m]{a^n}$$

(b) $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$ with

$$(a_1, a_2) \oplus (b_1, b_2) = (a_1 + a_2 - 2, b_1 + b_2 - 3)$$
$$\alpha \odot (a, b) = \alpha(a - 2, b - 3) + (2, 3)$$

(c) $V = \mathrm{M}_n(\mathbb{R})$, $\mathbb{F} = \mathbb{R}$ with

$$A \oplus B = 2A + 2B,$$
$$c \odot A = cA^T$$

2) (25 pts) Let V be an arbitrary real vector space. Prove that either V has one element or V has infinitely many elements.

3) (30 pts) Are the following subsets S of the vector space V subspaces? Justify.

(a) $S = S_1 \cap S_2$ for $S_1 = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ and $S_2 = \{(x, y, z) \in \mathbb{R}^3 : 2x - y + 3z = 0\}$ where $V = \mathbb{R}^3$.

(b) $S = \{A \in \mathrm{M}_n(\mathbb{R}) : A^T = A\}$ where $V = \mathrm{M}_n(\mathbb{R})$.

(c) $S = \{p(x) \in \mathbb{R}[x] : p(x)$ has only even powers of x$\}$ where $V = \mathbb{R}[x]$.