HOMEWORK 1

1) (30 pts) Decide if the following statements are true or false and justify your answer:

(a) Let $\mathbb{F}$ be a field and $a \in \mathbb{F}$. If there exists a natural number n such that $a^n = 0$ (where $a^n = a.a.\dots.a$ n times), then $a = 0$.

(b) Let $\mathbb{F}$ be a field and $a \in \mathbb{F}$. If there exists a natural number n such that $n.a = 0$ (where $n.a = a+a+\cdots+a$ n times), then $a = 0$.

(c) Let $\mathbb{F}$ be a field. If there exists $a \in \mathbb{F}$, $a \neq 0$ and a natural number n such that $n.a = 0$, then $n.x = 0$ for every $x \in \mathbb{F}$.

2) (20 pts) (30 pts) Consider $G_n = \{w \in \mathbb{C} \colon w \text{ is a n-th root of unity}\}$. Reduce the following expressions:

(a) $w^{22} + (w^2 + w)^2 + \overline{w}^{106} - w^{31} + w^{138}$ for any $w \in G_7$.

(b) $w^{-51} - \overline{w}^{74} + w^{33}.w^{-355} + \overline{w^{-127}}$ for any $w \in G_5$.

(c) $\displaystyle\sum_{k=8}^{19} w^{2k+1}$ for any $w \in G_{12}$.

3) (20 pts) Find a polynomial $p(x) \in \mathbb{R}[x]$ of minimal degree satisfying the following properties:

- The solutions of $z^2 + z = \overline{z}$ are roots of $p(x)$.
- The sum of all the roots of $p(x)$ is equal to zero.
- One of the real roots of $p(x)$ has multiplicity $m = 2$.

4) Let $p(x) = 6x^5 - 25x^4 + 39x^3 - 2x^2 - 6x$. Write $p(x)$ as a product of irreducible polynomials over the following fields:

(a) $\mathbb{R}$.

(b) $\mathbb{C}$.