Algebra A.

3.3. CLASS 1

USEFULL for what? The more you doubt, the more you learn.

Example: $f: R \to R^2$, $f(1) = (3,-1)$. can we deduce $f(a)$ for any $a \in R$?

$f: R \to R^2$ is a Function satisfying:

1) $f(1) = (3,-1)$

2) R and R^2 HAVE ALGEBLA STRUCTURES AND f behaves:

$R \times R \xrightarrow{\cdot} R$, $R \times R^2 \xrightarrow{*} R^2$.

$(a,b) \longrightarrow ab.$ $(a,(x,y)) \longrightarrow (ax, ay).$

$f(ab) = a \cdot f(b).$ $f(a) = (x_a, y_a)$ $\underset{(x_{ab}, y_{ab})}{\underset{\parallel}{f(ab)}} = a * f(b).$

$(x_{ab}, y_{ab}) = a*(x_b, y_b) = (ax_b, ay_b).$

$f(a) = f(a \cdot 1) = a * f(1) = a*(3,-1) = (3a, -a).$

F. VECTOR SPACES: sets of vectors with two Algebraic STRUCTURES.

LINEAR MAPS: Functions Between F. v.s. That behave well The Algebraic ST.

FIELD = SET WITH THE ALGEBRA STRUCTURES.

Field.

Def: A field is a nonempty set F with Two ALGEBRA STRUCTURES.

$+ : F \times F \to F.$ Addition $(a,b) \to a+b.$

$\cdot : F \times F \to F$ Product $(a,b) \to a \cdot b.$

SATISFYING THE FOLLOWING AXIOMS:

ASSOCIATIVITY: S_1: $(a+b)+c = a+(b+c).$ $\forall a, b \in F.$

P_1: $(a \cdot b) \cdot c = a(b \cdot c)$

COMMUTATIVITY: S_2: $a+b = b+a$ $\forall a, b \in F.$

P_2: $a \cdot b = b \cdot a$

IDENTITY: S_3: $\exists 0 \in F$: $a+0 = 0+a = a$ $\forall a \in F.$

P_3: $\exists 1 \in F$: $a \cdot 1 = 1 \cdot a = a$ $\forall a \in F.$

INVERSE: S_4: $\forall a \in F$. $\exists -a \in F$: $a + (-a) = (-a) + a = 0.$

P_4: $\forall a \in F$, $a \neq 0$, $\exists a^{-1} \in F$. $a \cdot a^{-1} = a^{-1} a = 1.$

DISTRIBUTIVITY: D: $a(b+c) = ab + ac$ $\forall a, b, c \in F.$

Example:

1) $N=\{1,2,3,4\cdots\}$ $\quad N\times N\xrightarrow{+}N$ (INDUCTION). $\quad S_1 S_2 \cancel{S_3} \cancel{S_4}$, D. ⇒ NOT A FIELD.

$\quad N\times N\xrightarrow{\cdot}N$ $\quad P_1 P_2 P_3 \cancel{P_4}$

2). $N_0=\{0,1,2,3,\cdots\}$ $\quad N_0\times N_0\xrightarrow{+}N_0$ $\quad S_1 S_2 S_3 \cancel{S_4}$ D. ⇒ NOT A FIELD.

$\quad N_0\times N_0\xrightarrow{\cdot}N_0$ $\quad P_1 P_2 P_3 \cancel{P_4}$

3) $Z=\{\cdots-2,-1,0,1,2\cdots\}$ $\quad Z\times Z\xrightarrow{+}Z$ $\quad S_1 S_2 S_3 S_4$ D. ⇒ NOT A FIELD.

$\quad Z\times Z\xrightarrow{\cdot}Z.$ $\quad P_1 P_2 P_3 \cancel{P_4}$

4) $Q=\{\frac{a}{b}, a,b\in Z, b\neq 0\}$, $\quad Q\times Q\xrightarrow{+}Q.$ $\quad S_1 S_2 S_3 S_4$ D. ⇒ IS A FIELD.

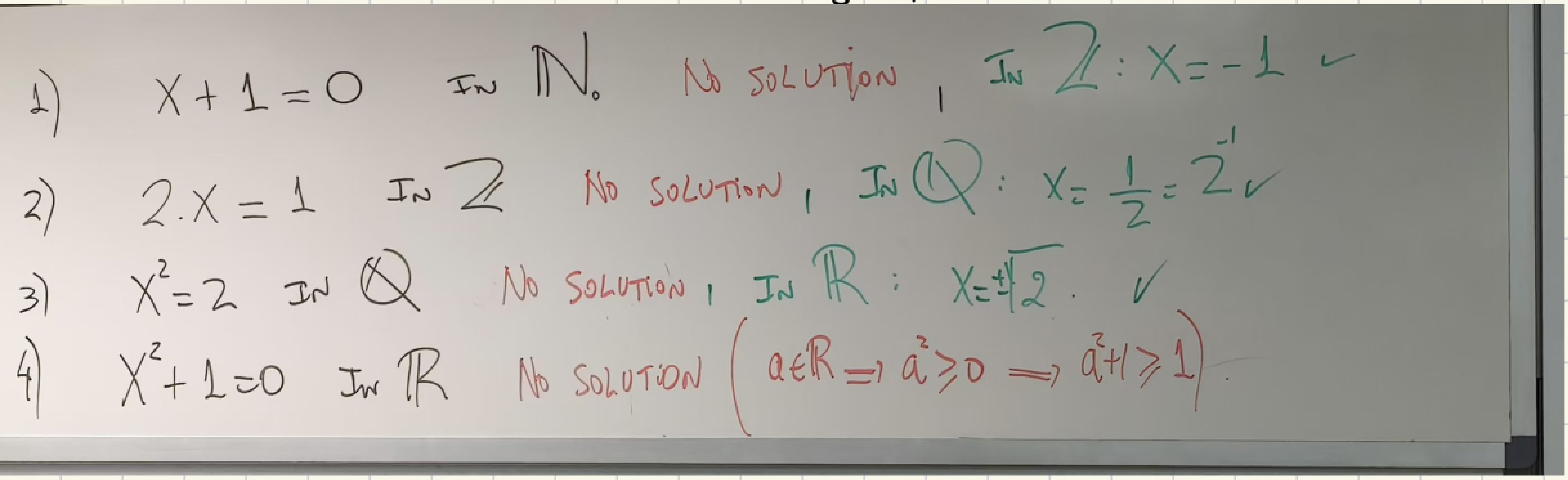
$\quad Q\times Q\xrightarrow{\cdot}Q.$ $\quad P_1 P_2 P_3 P_4$

I) R. $\quad R\times R\xrightarrow{+}R$

$\quad R\times R\xrightarrow{\cdot}R$ $\quad$ IS A FIELD.

$N\subseteq N_0\subseteq Z\subseteq Q\subseteq R\subseteq C.$

PROBLEMS: Find solutions for the following equations:



1) $X+1=0$ IN $\mathbb{N}_0$ NO SOLUTION, IN $\mathbb{Z}$: $X=-1$ ✓

2) $2.X=1$ IN $\mathbb{Z}$ NO SOLUTION, IN $\mathbb{Q}$: $X=\frac{1}{2}=2^{-1}$ ✓

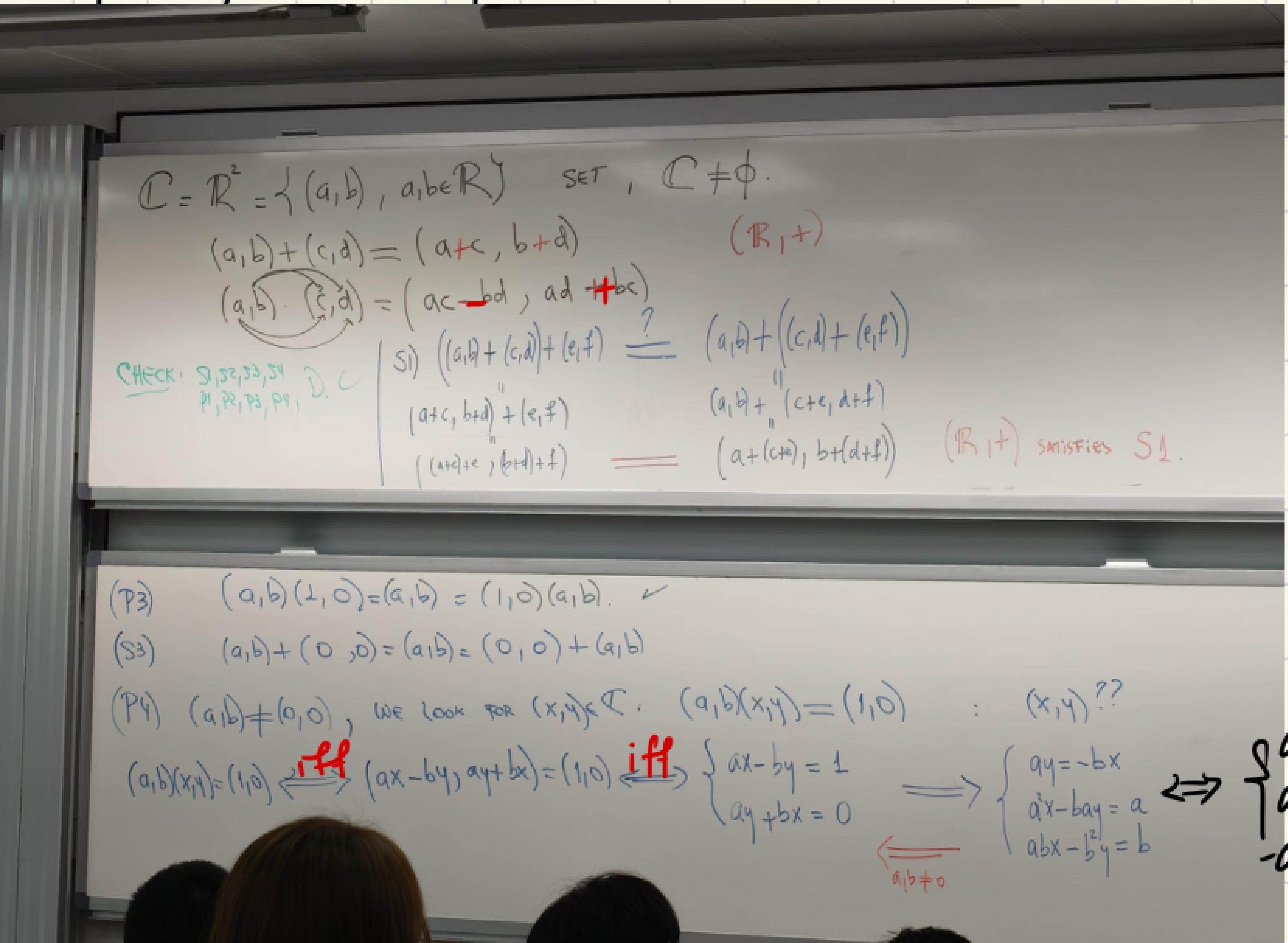
3) $X^2=2$ IN $\mathbb{Q}$ NO SOLUTION, IN $\mathbb{R}$: $X=\pm\sqrt{2}$. ✓

4) $X^2+1=0$ IN $\mathbb{R}$ NO SOLUTION $\left(a\in\mathbb{R}\Rightarrow a^2\geq 0\Rightarrow a^2+1\geq 1\right)$.

C = Set of COMPLEX NUMBERS.

PROPERTY:

Any polynomial equation in C that has a solution in C.



$\mathbb{C}=\mathbb{R}^2=\{(a,b), a,b\in\mathbb{R}\}$ SET, $\mathbb{C}\neq\phi$.

$(a,b)+(c,d)=(a+c, b+d)$ $\quad (\mathbb{R},+)$

$(a,b)\cdot(c,d)=(ac-bd, ad+bc)$

CHECK: S1,S2,S3,S4 P1,P2,P3,P4, D. C

S1) $((a,b)+(c,d))+(e,f)\overset{?}{=}(a,b)+((c,d)+(e,f))$

$(a+c,b+d)+(e,f)$ $\quad (a,b)+(c+e,d+f)$

$((a+c)+e,(b+d)+f) = (a+(c+e), b+(d+f))$ $\quad (\mathbb{R},+)$ SATISFIES S1.

(P3) $(a,b)(1,0)=(a,b)=(1,0)(a,b).$ ✓

(S3) $(a,b)+(0,0)=(a,b)=(0,0)+(a,b)$

(P4) $(a,b)\neq(0,0)$, WE LOOK FOR $(x,y)\in\mathbb{C}$: $(a,b)(x,y)=(1,0)$: (x,y)??

$(a,b)(x,y)=(1,0)\overset{\text{iff}}{\Longleftrightarrow}(ax-by, ay+bx)=(1,0)\overset{\text{iff}}{\Longleftrightarrow}\begin{cases}ax-by=1\\ay+bx=0\end{cases}\underset{a,b\neq 0}{\Longleftrightarrow}\begin{cases}ay=-bx\\a^2x-bay=a\\abx-b^2y=b\end{cases}$

We can check if:

$(a,b)\cdot\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)=(1.0)$ ✓.

$a^2+b^2\neq 0$ since $(a,b)\neq(0,0)$.

$\Leftrightarrow\begin{cases}ay=-bx\\a^2x-b^2x=a\\-a^2y-b^2y=b\end{cases}\Leftrightarrow(x,y)=\left(\frac{a}{a^2+b^2},\frac{-b}{a^2+b^2}\right)$

C: R^2 is a Field.

$R \rightarrow R^2$ INJECTIVE MAP.

$a \mapsto (a,0)$

$(a,0)+(b,0) = (a+b,0+0) = (a+b,0)$. BEHAVE WEll WITH ADDICTION. PRODUCT.

$(a,0)\cdot(b,0) = (ab-00, a\cdot 0+0b) = (ab,0)$

$(a,b) = (a,0)+(b,0) = (a,0)+(b,0)(0,1) = a+b(0,1) = a+bi$ Binomial Expression.

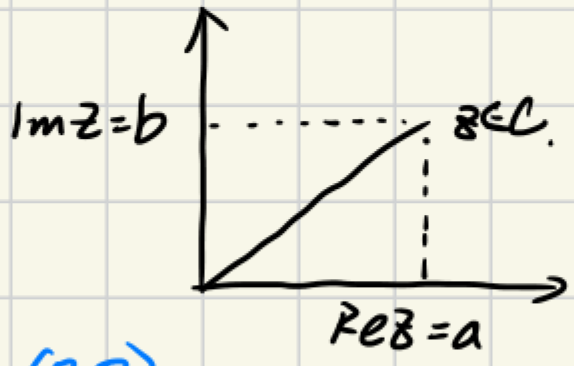
Remark:

1) $i^2 = (0,1)(0,1) = (-1,0) = -1$.

2) $(a+bi)(c+di) = (a+c)+(b+d)i$.

3) $(a+bi)\cdot(c+di) = a(c+di)+bi(c+di) = (ac-bd)+(ad+bc)i$.

DEFINITION: $z = (a,b) = a+bi \in C$.



$a = \mathrm{Re}\,z$ = Real part of z. $b = \mathrm{Im}\,z$ = Imaginary part of z.

$\bar{z} = (a,-b) = a-bi$ CONJUGATE of z.

$|z| = \sqrt{a^2+b^2} \in R_{\geq 0}$. MODULE of z. $\Leftarrow |z|$ = distance of (a,b) to $(0,0)$.

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Def: let $z = (a,b) = a+bi \in C$.

1) MODULE: $|z| = \sqrt{a^2+b^2}$. = DISTANCE $((a,b),(0,0))$.

2) CONJUGATE: $\bar{z} = a-bi$

PROPERTIES ABOUT CONJUGATE:

1) $\bar{\bar{z}} = z$.

2) $z+\bar{z} = 2\mathrm{Re}\,z$.

3) $z-\bar{z} = 2\mathrm{Im}\,z\cdot i$

4) $z = \bar{z} \Longleftrightarrow z = \mathrm{Re}\,z \in R$.

5) $\overline{z+w} = \bar{z}+\bar{w}$ $\forall z,w \in C$.

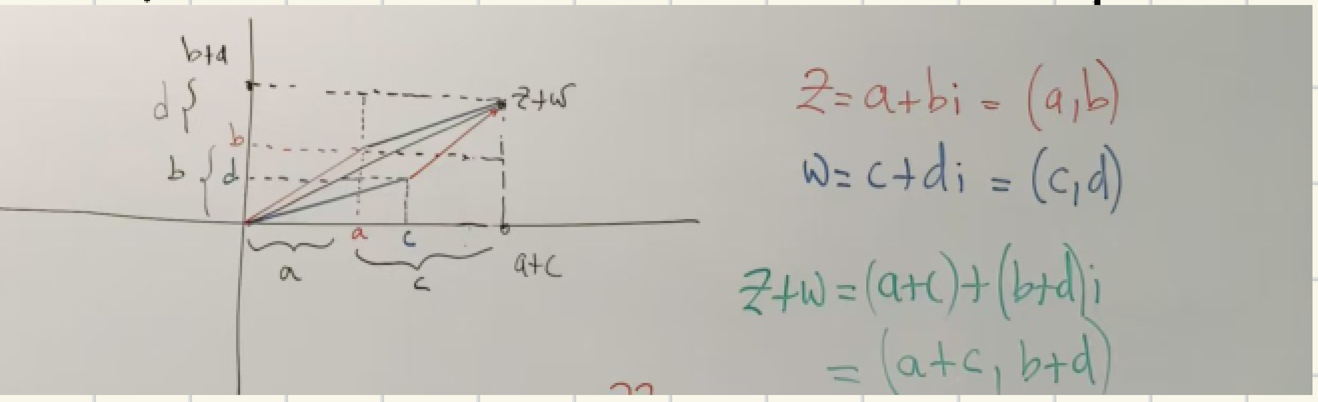
6) $\overline{zw} = \bar{z}\cdot\bar{w}$.

Proof: let $z = a+bi$ $w = c+di$

5) $\overline{z+w} = \overline{a+bi+c+di} = \overline{a+c+(b+d)i} = a+c-(b+d)i \Rightarrow \overline{z+w} = \bar{z}+\bar{w}$.

$\bar{z}+\bar{w} = \overline{a+bi}+\overline{c+di} = a-bi+c-di = a+c-(b+d)i$.

GEOMETRIC INTERPRETATION of $z+w$.



$z = a+bi = (a,b)$

$w = c+di = (c,d)$

$z+w = (a+c)+(b+d)i$

$= (a+c, b+d)$

PROPERTIES OF MODULE:

1) $z\cdot\bar{z} = |z|^2$.

2) $|z| = |-z| = |\bar{z}|$

3) $|z\cdot w| = |z|\cdot|w|$.

4) $\frac{|z|}{|w|} = \left|\frac{z}{w}\right|$, $w \neq 0$. $(|z+w| \neq |z|+|w|)$.

Proof:

(1) $z\cdot\bar{z}=(a+bi)(a-bi)=a^2+bai-abi-b^2i^2=a^2+b^2=(\sqrt{a^2+b^2})^2=|z|^2$.

To prove (3), we should prove the lemma: $x=y \Leftrightarrow x^2=y^2$ for $x,y\geq 0$ $x,y\in\mathbb{R}$.

$x^2=y^2 \Leftrightarrow x^2-y^2=0 \Leftrightarrow (x+y)(x-y)=0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$. ⊠

So By the DEFINITION, $|z|=\sqrt{a^2+b^2}\in\mathbb{R}$, and $|z|\geq 0$. $|z\cdot w|=|z|\cdot|w| \Leftrightarrow |z\cdot w|^2=(|z|\cdot|w|)^2$.

$|z\cdot w|^2=(zw)\overline{(zw)}=(zw)\cdot(\bar{z}\cdot\bar{w})=(z\cdot\bar{z})(w\cdot\bar{w})=|z|^2\cdot|w|^2=(|z|\cdot|w|)^2$ ⊠

Remark:

(1) CONJUGATE of COMPLEX NUMBERS extends the absolute value of REAL NUMBERS.

$z=(a,0)=a+0i$, $|z|=\sqrt{a^2+0^2}=\sqrt{a^2}=|a|$

(2)

$z=a+bi\neq 0 \Rightarrow z^{-1}=\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i$

Since $z\cdot\bar{z}=|z|^2=a^2+b^2\in\mathbb{R}_{>0}$. $\Rightarrow$ $z\cdot\bar{z}\cdot\frac{1}{a^2+b^2}=1$. So $z^{-1}=\bar{z}\cdot\frac{1}{a^2+b^2}$.

So $z^{-1}=(a-bi)\cdot\frac{1}{a^2+b^2}=\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i$.

PROPERTIES:

1) $\mathrm{Re}z\leq|\mathrm{Re}z|\leq|z|$.

2) $|z+w|\leq|z|+|w|$ TRIANGLE

LAMMA: $x,y\in\mathbb{R}$, $x,y\geq 0$ then $x\geq y\Leftrightarrow x^2\geq y^2$.

Proof: $x^2\geq y^2\Leftrightarrow x^2-y^2\geq 0\Leftrightarrow (x+y)(x-y)\geq 0$ since $x+y\geq 0$

So $x-y\geq 0\Leftrightarrow x\geq y$.

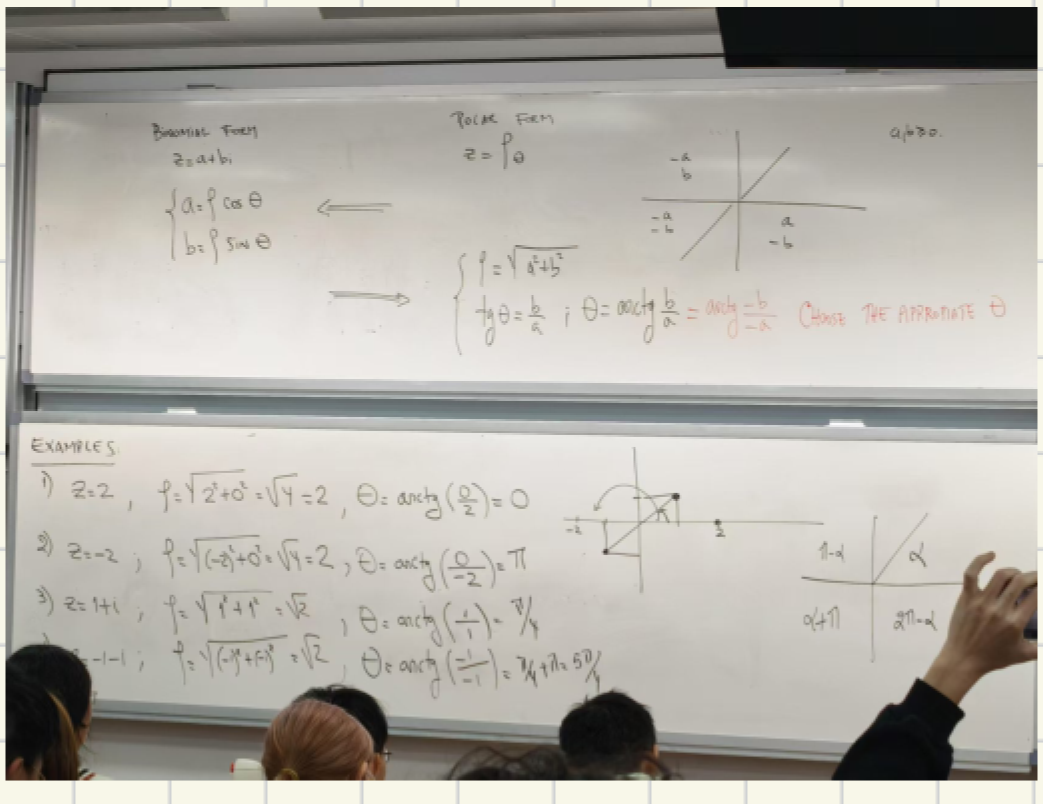
Remark In particular: $\sqrt{x}\geq\sqrt{y}\Leftrightarrow x\geq y\geq 0$.

Proof:

1) $z=a+bi\Rightarrow a\leq|a|=\sqrt{a^2}\leq\sqrt{a^2+b^2}$. since $a^2\leq a^2+b^2$.

$\Rightarrow \mathrm{Re}z\leq|\mathrm{Re}z|\leq|z|$

2) $|z+w|\leq|z|+|w| \Leftrightarrow |z+w|^2\leq(|z|+|w|)^2$.



BINOMIAL FORM

$z=a+bi$

$\begin{cases}a=\rho\cos\theta\\ b=\rho\sin\theta\end{cases}$ $\Longleftarrow$

POLAR FORM

$z=\rho_\theta$

$a,b>0$.

$\Longrightarrow$ $\begin{cases}\rho=\sqrt{a^2+b^2}\\ tg\theta=\frac{b}{a}\ ;\ \theta=arctg\frac{b}{a}=arctg\frac{-b}{-a}\end{cases}$ CHOOSE THE APPROPRIATE θ

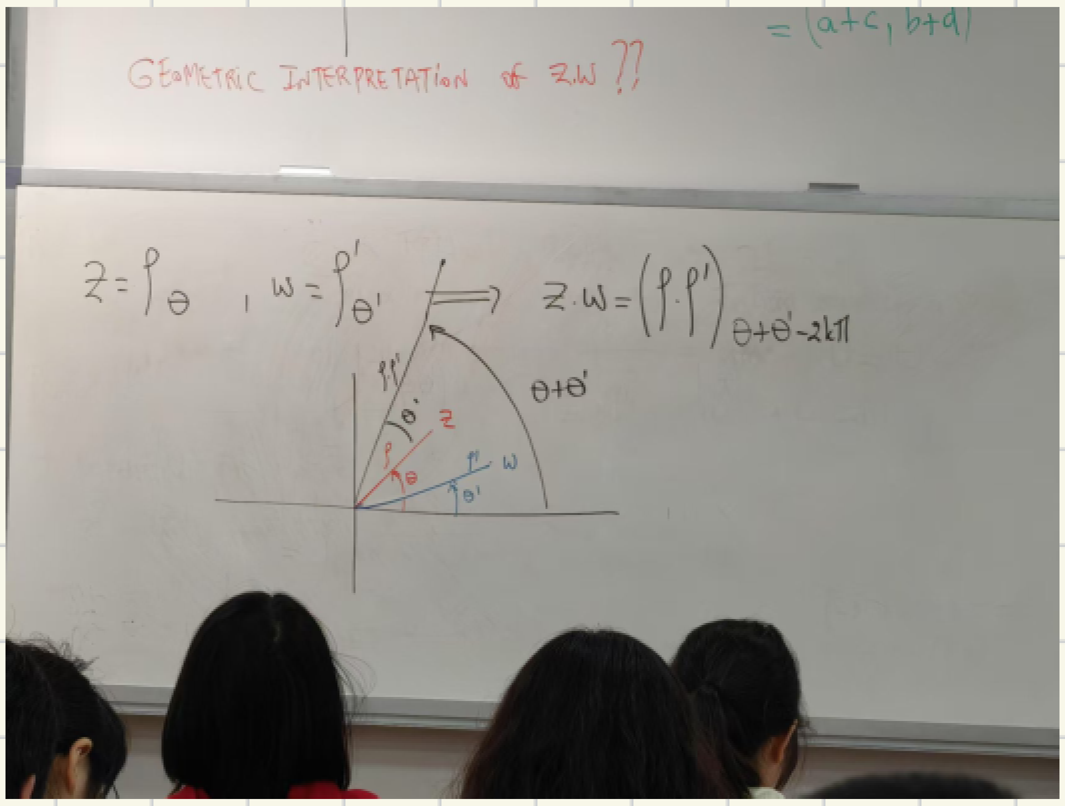
EXAMPLES

1) $z=2$, $\rho=\sqrt{2^2+0^2}=\sqrt{4}=2$, $\theta=arctg\left(\frac{0}{2}\right)=0$

2) $z=-2$; $\rho=\sqrt{(-2)^2+0^2}=\sqrt{4}=2$, $\theta=arctg\left(\frac{0}{-2}\right)=\pi$

3) $z=1+i$; $\rho=\sqrt{1^2+1^2}=\sqrt{2}$, $\theta=arctg\left(\frac{1}{1}\right)=\pi/4$

$-1-i$; $\rho=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$, $\theta=arctg\left(\frac{-1}{-1}\right)=\pi/4+\pi=5\pi/4$



$=(a+c, b+d)$

GEOMETRIC INTERPRETATION of Z.W ??

$z=\rho_\theta$, $w=\rho'_{\theta'}$ $\Longrightarrow$ $z\cdot w=(\rho\cdot\rho')_{\theta+\theta'-2k\pi}$

3.10. CLASS 3.

BINOMIAL FORM. $z=a+bi=(a,b)$.

$(a,b)+(c,d)=(a+c,b+d)$.

$(a,b)(c,d)=(ac-bd,\ ad+bc)$.

$a\cdot(b,c)=(a,0)\cdot(b,c)=(ab-0\cdot c,\ ac+0\cdot b)=(ab,ac)$.

POLAR FORM

$z=P_\theta$, $P=|z|$ $\theta=\mathrm{Arg}z$. if $z=a+bi$. $P=\sqrt{a^2+b^2}$. $\theta=\mathrm{Arg}(\frac{b}{a})$. $(\theta\in 0,2\pi)$.

$\begin{cases} a=P\cos\theta. \\ b=P\sin\theta. \end{cases} \longrightarrow z=P_\theta$.

$z=P(\cos\theta+i\sin\theta)=P\,\mathrm{cis}\,\theta=P\cdot e^{i\theta}$.

Remark: if $z=0 \Leftrightarrow |z|=0$ POLAR FORM FOR $z=0$ IS $P=0$. θ is not define.

POLAR FORM FOR PRODUCT.

Theorem: If $z,w\in\mathbb{C}$, $z,w\neq 0$. Then the polar form of zw is given by:

$|zw|=|z|\cdot|w|$.

$\mathrm{Arg}(zw)=\mathrm{Arg}z+\mathrm{Arg}w,-2k\pi$. For $k\in\mathbb{Z}$.

PROOF:

$z\cdot w=(P_z\cos(\mathrm{Arg}z)+P_z\sin(\mathrm{Arg}z)i)(P_w\cos(\mathrm{Arg}w)+P_w\sin(\mathrm{Arg}w)\cdot i)$.

$=P_z\cdot P_w\cdot(\cos(\mathrm{Arg}z+\mathrm{Arg}w)+i\sin(\mathrm{Arg}z+\mathrm{Arg}w))$.

COROLLARY: If $z\in\mathbb{C}$, $n\in\mathbb{N}$ $\Rightarrow$ the polar form of z^n is:

$|z^n|=|z|^n$.

$\mathrm{Arg}(z^n)=n\cdot\mathrm{Arg}(z)-2k\pi$. for $k\in\mathbb{Z}$. $\Rightarrow$ easy to prove by induction.

Example: $z=(1+i)^{1342}$.

Since $|1+i|=\sqrt{2}$. $\Rightarrow$ $1+i=\sqrt{2}\cdot e^{i\frac{\pi}{4}}$

So $|z|=|(1+i)^{1342}|=|1+i|^{1342}=(\sqrt{2})^{1342}=2^{671}$ $\mathrm{Arg}(z)=1342\cdot\mathrm{Arg}(1+i)+2k\pi, k\in\mathbb{Z}$.

$\mathrm{Arg}(z)=1342\cdot\frac{\pi}{4}+2k\pi=\frac{(671-4k)\pi}{2}\in[0,2\pi)$.

Now, we need to compute k: $0\leq\frac{(671-4k)}{2}\pi<2\pi$. $\Leftrightarrow$ $0\leq 671-4k<4$.

This is a RIMINDER of divition by 4.

$671=4\cdot 167+3$. So $\mathrm{Arg}(z)=\frac{(671-4k)\pi}{2}=\frac{3}{2}\pi$. So $z=2^{671}e^{i\frac{3}{2}\pi}=-2^{671}i$.

ROOT.

PROBLEM: we want to find all $z\in\mathbb{C}$. that satisfy $z^n=w$. For some $w\in\mathbb{C}$ $n\in\mathbb{N}$.

REMARK: If $w=0$. $\Rightarrow z^n=0 \Leftrightarrow |z|^n=|z^n|=0 \Leftrightarrow |z|=0. \Leftrightarrow z=0$.

THEO: Let $w\in\mathbb{C}$, $w\neq 0$. $n\in\mathbb{N}$ then:

$z^n=w \Rightarrow z=\sqrt[n]{|w|}\cdot(\cos\frac{\mathrm{Arg}w+2k\pi}{n}+i\sin\frac{\mathrm{Arg}w+2k\pi}{2})$. $k=0,1,2\cdots n-1$

Proof: $z^n = w \Leftrightarrow |z^n| = |w|$ And $\text{Arg}(z^n) = \text{Arg}\,w$. $\Leftrightarrow |z|^n = |w|$ And $n\cdot\text{Arg}\,z - 2k\pi = \text{Arg}\,w \in [0,2\pi)$.

$\Leftrightarrow |z| = \sqrt[n]{|w|}$ = unique positive real n-root of the positive real number $|w|$.

And $\text{Arg}\,z = \frac{\text{Arg}\,w + 2k\pi}{n} \in [0,2\pi)$, $k\in\mathbb{Z}$.

$\Rightarrow 0 \le \frac{\text{Arg}\,w+2k\pi}{n} < 2\pi$. $\Leftrightarrow 0 \le \text{Arg}\,w + 2k\pi < 2n\pi$.

$\Rightarrow -\text{Arg}\,w \le 2k\pi < 2n\pi - \text{Arg}\,w$. since $0 \le \text{Arg}\,w < 2\pi$. $\Rightarrow -2\pi < -\text{Arg}\,w \le 0$.

$\Leftrightarrow -2\pi < -\text{Arg}\,w \le 2k\pi < 2n\pi - \text{Arg}\,w \le 2n\pi - 0$.

$\Leftrightarrow -2\pi < 2k\pi < 2n\pi \Leftrightarrow -1 < k < n$. $\Leftrightarrow k = 0,1,2,3,\cdots n-1$

Example: For all $z\in\mathbb{C}$. st $z^4 = 16$.

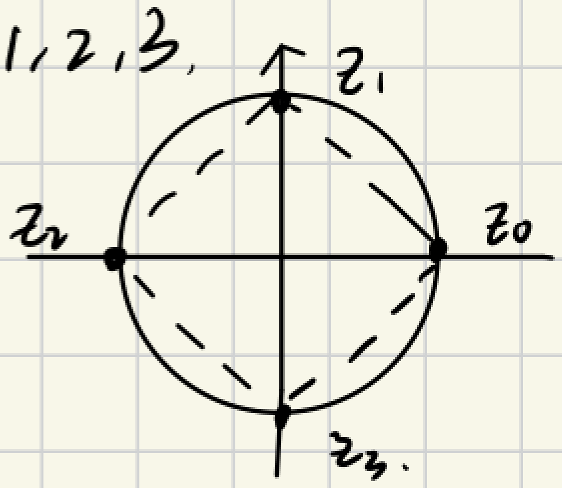
$z_k = \sqrt[4]{16}\,\text{cis}\left(\frac{0+2k\pi}{4}\right)$. $k = 0,1,2,3,$

$z_0 = 2\cdot\text{cis}\,0 = 2$.

$z_1 = 2\cdot\text{cis}\frac{\pi}{2} = 2i$

$z_2 = 2\cdot\text{cis}\,\pi = -2$.

$z_3 = 2\cdot\text{cis}\frac{3\pi}{2} = -2i$.

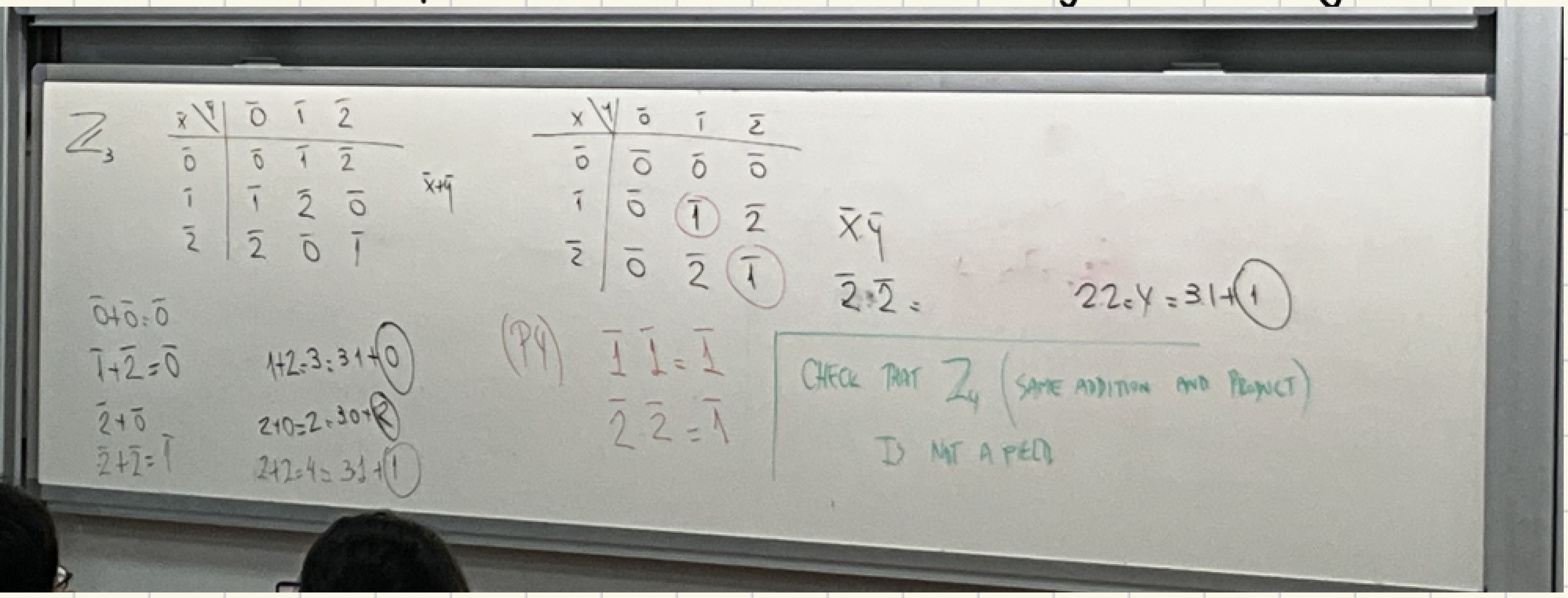


F = FIELDS.

EXAMPLES: $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

OTHER EXAMPLES: $\mathbb{Q}[i] = \{a+bi,\ a,b\in\mathbb{Q}\}$. $\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2},\ a,b\in\mathbb{Q}\}$.

Finite field: $\mathbb{Z}_p = \{\bar{0},\bar{1},\bar{2},\bar{3}\dots\overline{p-1}\}$, $\bar{x}+\bar{y} = \bar{r}$. $\bar{x}\bar{y} = \bar{r'}$.



$\mathbb{Z}_3$

$\bar{x}\backslash\bar{y}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

$\bar{x}+\bar{y}$

$x\backslash y$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\textcircled{\bar{1}}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\textcircled{\bar{1}}$

$\bar{x}\bar{y}$

$\bar{2}\cdot\bar{2} =$

$2\cdot2 = 4 = 3\cdot1 + \textcircled{1}$

$\bar{0}+\bar{0} = \bar{0}$

$\bar{1}+\bar{2} = \bar{0}$ $\quad 1+2 = 3 = 3\cdot1+\textcircled{0}$

$\bar{2}+\bar{0}$ $\quad 2+0 = 2 = 3\cdot0+\textcircled{2}$

$\bar{2}+\bar{2} = \bar{1}$ $\quad 2+2 = 4 = 3\cdot1+\textcircled{1}$

(P4) $\bar{1}\,\bar{1} = \bar{1}$

$\bar{2}\,\bar{2} = \bar{1}$

CHECK THAT $\mathbb{Z}_4$ (SAME ADDITION AND PRODUCT) IS NOT A FIELD

POLYNOMIALS ON ONE VARIABLE X WITH COEFFICIENTS ON A FIELD F.

$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, $a_i \in F$, $n\in\mathbb{N}$. x is an Indeterminate.

$= (a_0, a_1, a_2, a_3 \dots a_n, 0, 0, \dots)$

DEFINITIONS $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n = g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots + b_mx^m$.

$\Leftrightarrow a_k = b_k$. $\forall k \ge 0$.

Example: (1) $f(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n = 0 = 0 + 0x + 0x^2 + \dots + 0x^n$.

$(a_0, a_1, a_2, a_3 \cdots a_n, 0, 0 \dots)$ $\quad (0,0,0,0,0\dots)$

$\Leftrightarrow a_0 = a_1 = a_2 \dots = a_n = 0$.

(2) $f(x) = a_0x + a_1x^2 + a_2x^3 + \dots + a_nx^n = 0 \Leftrightarrow a_k = 0\ \forall k \ge 0$.

(3) $F[x]$ = set of All POLYNOMIALS IN ONE VARIABLE WITH COEFFICENTS In F.

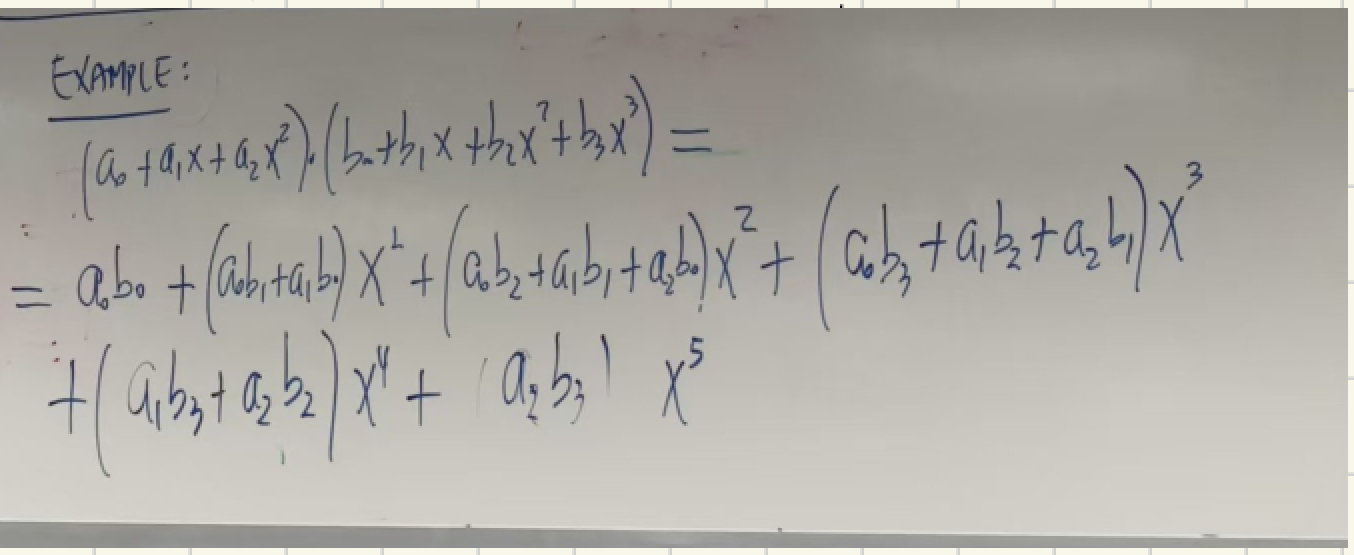
(4) We can define an ADDITION And a PRODUCT In $F(x)$.

$F(x) \times F(x) \xrightarrow{+} F(x)$.

$\sum a_i x^i + \sum b_i x^i = \sum (a_i+b_i) x^i$

$F(x) \times F(x) \xrightarrow{\times} F(x)$

$\sum a_i x^i \times \sum b_i x^i = \sum \left(\sum_{i+j=k} a_i b_j\right) x^k$.



EXAMPLE:

$(a_0+a_1x+a_2x^2)\cdot(b_0+b_1x+b_2x^2+b_3x^3) =$

$= a_0b_0 + (a_0b_1+a_1b_0)x^1 + (a_0b_2+a_1b_1+a_2b_0)x^2 + (a_0b_3+a_1b_2+a_2b_1)x^3$

$+ (a_1b_3+a_2b_2)x^4 + (a_2b_3)x^5$

$(F(x), +, \cdot)$ satisfies S_1, S_2, S_3, S_4. D.

P_1 P_2, P_3, ~~P_4~~ (Check that these is no $f(x)$ st $f(x)\cdot x = 1$)

$\Downarrow$

NOT A FIELD.

REMARK: Compare with $(\mathbb{Z}, +, \cdot)$ Without P_4, we called it RINGS.

DEF: Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in F(x)$ $f(x) \neq 0$.

1) $\deg f(x) = \text{Max}\{k, a_k \neq 0\}$.
2) If $n = \deg f(x)$, a_n = LEADING COEFFICIENT.
3) If $n = \deg f(x)$, $a_n = 1$ $f(x)$ is called MONIC.

PROPOSITION: If $f(x), g(x) \in F[x]$, $f(x) \cdot g(x) \neq 0$, then:

1) $f(x) \cdot g(x) \neq 0$.
2) $\deg(f(x) \cdot g(x)) = \deg f(x) + \deg g(x)$.
3) If $f(x) + g(x) \neq 0$ then $\deg(f(x) + g(x)) \leq \max\{\deg f(x), \deg(g(x))\}$.

3.12. CLASS 4.

In $\mathbb{Z}$, Any $n \in \mathbb{Z}$, $n \neq 0, 1, -1$. can be written in a UNIQUE way as (± 1) times a product of positive PRIME NUMBER.

$(\mathbb{Z}, +, \cdot)$ RING ← Positive Prime numbers.

$(F[x], +, \cdot)$ RING ← Monic IRREDUCIBLE POLYNOMIALS.

Def: A polynomial $f(x) \in F[x]$ is called IRREDUCIBLE if $f(x) \neq 0$, $\deg(f(x)) \neq 0$. And if $f(x) = g(x) \cdot h(x)$, $g, h \in F[x]$. $\Rightarrow \deg(g(x)) = 0$ or $\deg(h(x)) = 0$.

Remark: $f(x)$ is IRREDUCIBLE $\Leftrightarrow$ $f(x) \neq 0$, and it cannot be written as the product of two non-constant polynomials.

Example: $x^2 - 1 = (x+1)(x-1)$ NOT IRREDUCIBLE

$x - a = g(x) \cdot h(x) \Rightarrow \deg(x-a) = 1 = \deg(g(x)) + \deg(h(x)) \Rightarrow g(x) = 0$ or $h(x) = 0$.

So $g(x)$ or $h(x)$ is constant, $x-a$ is IRREDUCIBLE.

$x^2+1=(x-i)(x+i)$ NOT IRREDUCIBLE in $C[x]$

x^2+1 is IRREDUCIBLE In $Q[x]$ and $R[x]$.

$x^2-3=(x-\sqrt{3})(x+\sqrt{3})$ NOT IRREDUCIBLE in $R[x]$ or in $C[x]$.

x^2-3 is IRREDUCIBLE in $Q[x]$.

THEO: Any non-constant polynomial in $F[x]$ can be written in a UNIQUE As a constant times of Monic IRREDUCIBLE POLYNOMIALS.

(USE INDUCTION TO PROVE BY $\deg f(x)$).

Remark: IT IS NOT EASY TO DETERMINE IF A POLYNOMIALS $f(x)$ IS IRREDUCIBLE OR NOT.
AND IT IS NOT EASY TO FACTORIZE A POLYNOMIAL INTO A PRODUCT.

THE EASY IDEA IS FIND ROOTS.

DIVISION ALGORITHM IN F[x].

for any $f(x), g(x) \in F[x]$, $g(x) \neq 0$, these exist UNIQUE $q(x), r(x) \in F[x]$ st
$f(x)=g(x)\cdot q(x)+r(x)$, $r(x)=0$ or $\deg(r)<\deg(g)$.

Proof: $X=\{f(x)-g(x)q(x),\ q,g\in F[x]\}$. If $0\in X \Rightarrow 0=f(x)-g(x)\cdot q(x)$, $r(x)=0$ ✓.
if $0\notin X$, $\phi \neq \deg(X)=\{\deg(f(x)-q(x)\cdot g(x)),\ q,g\in F[x]\}\subseteq N_0$. WOP: $\exists s$ first element in $\deg X$.
$s=\deg(f(x)-g(x)q(x))$ Suppose $\deg(r)\geqslant\deg(g)$ ---- Contradiction.

Definition: let $f(x)\in F[x]$, $a\in F$, say that a is a root of $f(x)$ if $f(a)=0$.

Example: 2 is the root of x^3-3x-2 since $2^3-3\cdot2-2=0$.
1 is not root of x^3-3x-2 since $1^3-3\cdot1-2=-4\neq0$.

REMAINDER THEOREM

let $f(x)\in F[x]$, $a\in F$. the remainder of the division of $f(x)$ by $x-a$ is $f(a)$

Proof:

By the DIVISION ALGORITHM. $\exists!\, q(x), r(x)\in F[x]$: $f(x)=(x-a)\cdot q(x)+r(x)$ with
$r(x)=0$ or $\deg(r)<\deg(x-a)=1 \Rightarrow \deg(r)=0 \Rightarrow r(x)=c \Rightarrow$ constant.
$f(x)=(x-a)q(x)+c$. $f(a)=0\cdot q(a)+c \Rightarrow c=f(a) \Rightarrow f(x)=(x-a)q(x)+f(a)$. ∎

COROLLARY: let $f(x)\in F[x]$, $f(x)\neq0$. THEN a is a root of $f(x) \Leftrightarrow f(a)=0 \Leftrightarrow f(x)=(x-a)g$

3.7. CLASS 5.

VECTOR SPACES.

Def: Let F be a field, V be a set, then V is an F-vector space. If there are two operations:

ADDITION OF VECTORS: $+ : V \times V \longrightarrow V$.
$(v, w) \rightarrow v + w$.

PRODUCT BY SCALARS: $\cdot : F \times V \longrightarrow V$.
$(\lambda, v) \longrightarrow \lambda v$.

satisfying:

S1) **ASSOCIATIVITY**: $(v+w)+u = v+(w+u)$. $\forall v, w, u \in V$.

S2) **COMMUTATIVITY**: $v+w = w+v$, $\forall v, w \in V$.

S3) **IDENTITY**: $\exists 0 \in V$: $v + 0_v = v = 0_v + v$, $\forall v \in V$

S4) **INVERSE**: $\forall v \in V$. $\exists -v \in V$, $v + (-v) = 0_v = (-v) + v$.

M1) $(\lambda + \mu)\cdot v = \lambda \cdot v + \mu \cdot v$, $\forall \lambda, \mu \in F$. $v \in V$

M2) $(\lambda \cdot \mu)\cdot v = \lambda(\mu \cdot v)$, $\forall \lambda, \mu \in F$, $v \in V$.

M3) $\lambda \cdot (v+w) = \lambda \cdot v + \lambda \cdot w$ $\forall \lambda \in F$. $v, w \in V$.

M4) $1 \cdot v = v$ $\forall v \in V$.

V = SET OF VECTORS.
F = SET OF SCALARS.

Question: If (M4) say that $1 \cdot v = v$, why we say nothing about $0 \cdot v$?

Since $\underset{\text{from } F}{0} \cdot v = \underset{\text{from } V}{0}$.

Properties: Let V be an F-vector space, $v, w, u \in V$. $\lambda \in F$.

1) $v + u = w + u \Rightarrow v = w$.
$u + v = u + w \Rightarrow v = w$.

2) $0_F \cdot v = 0_v$; $\lambda \cdot 0_v = 0_v$.

3) $\lambda \cdot v = 0_v \Rightarrow \lambda = 0_F$ or $v = 0_v$.

4) $-v = (-1_F) \cdot v$.

Proof:

1) $v + u = w + u \xRightarrow{S_4} (v+u) + (-u) = (w+u) + (-u)$ since $-u \in V \xRightarrow{S_1} v + (u-u) = w + (u-u)$.
$\xRightarrow{S_4} v + 0_v = w + 0_v \xRightarrow{S_3} v = w$. $u + v = u + w \Rightarrow v = w$ use the same way. ∎

2) $0_F \cdot v + 0_v \overset{S_3}{=} 0_F \cdot v \overset{S_3}{=} (0_F + 0_F)\cdot v \overset{M_1}{=} 0_F \cdot v + 0_F \cdot v \Rightarrow 0_v = 0_F \cdot v$. ∎
$\lambda \cdot 0_v + 0_v \overset{S_3}{=} \lambda \cdot 0_v \overset{S_3}{=} \lambda \cdot (0_v + 0_v) \overset{M_3}{=} \lambda 0_v + \lambda 0_v \Rightarrow \lambda \cdot 0_v = 0_v$.

3) $\lambda \cdot v = 0_v$ we want to see that $\lambda = 0_F$ or $v = 0_v$. If $\lambda = 0_F$. we done,
If $\lambda \neq 0_F$. By P4. $\exists \lambda^{-1} \in F$. $0_v \overset{2}{=} \lambda^{-1} \cdot 0_v = \lambda^{-1} \cdot (\lambda \cdot v) = (\lambda^{-1} \cdot \lambda) \cdot v = 1_F \cdot v = v$.
$\Rightarrow a = v$.

4) $-v = (-1_F)v \iff v + (-1)\cdot v = 0_v$ Since $v+(-1)v \overset{M_4}{=} 1\cdot v + (-1)v \overset{M_1}{=} (1+(-1))v = 0_F \cdot v \overset{(2)}{=} 0_v$.

MORE PROPETIES:

1) 0_v is unique.

2) For any $v \in V$, $-v$ is unique.

Proof:

(1) If $0, 0'$ are IDENTITIES In V satisfying (S_3), then:

$$\underset{0 \text{ identity}}{0' \overset{}{=}} 0 + 0' \underset{0' \text{ identity}}{=} 0.$$

we have done.

(2) If v_1, v_2 are INVERSES OF v. then:

$$v_1 = v_1 + 0_v = v_1 + (v + v_2) \overset{S_1}{=} (v_1 + v) + v_2 = 0_v + v_2 = v_2.$$

$\Rightarrow v_1 = v_2$ ☒

Remark: $(S_3) \Rightarrow V \neq \phi$ since $0_v \in V$.

Example:

1) $V = \{*\}$ a set with only one element in F-vector space.

$V \times V \to V$. $\quad F \times V \to V$.

$(*, *) \to * + *$. $\quad (\lambda, *) \to \lambda \cdot *$.

and satisfies $S_1 \to S_4$, $M_1 \to M_4$.

2) $V = F$ is an F-vector space.

$(F, +, \cdot)$ field $\quad (\underset{\text{vector}}{F},\ \underset{\text{ADDITION OF VECTOR.}}{F \times F \xrightarrow{+} F},\ \underset{\text{PRODUCT OF VECTORS}}{F \times F \xrightarrow{\cdot} F})$ VECTOR SPACE.

$+ = +$ $\quad \cdot = \cdot$

$S_1 = S_1$ $\quad S_2 = S_2$ $\quad (S_3): 0_v = 0_F$, $\quad S_4 = S_4$ $\quad M_1 = D$ $\quad M_2 = P_1$ $\quad M_3 = D$ $\quad M_4 = P_3$.

$\mathbb{R}$ is an $\mathbb{R}$-VECTORS SPACE.

$\mathbb{C}$ is an $\mathbb{C}$-VECTORS SPACE.

3) $(F, +, \cdot)$ FIELD, $\quad V = F \times F = \{(x, y) : x, y \in F\}$.

$(x, y) + (x', y') = (x + x', y + y')$

$\lambda(x, y) = (\lambda \cdot x, \lambda \cdot y)$

S_2. $(x, y) + (x', y') = (x + x', y + y') = (x' + x, y' + y) = (x', y') + (x, y)$.

S_3. $0_v = (0, 0)$ since $(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$.

S_4. $-(x, y) = (-x, -y)$.

4) $(F, +, \cdot)$ FIELD, $\quad V = F^n = \underbrace{F \times F \times F \times F \times \cdots F}_{\text{for } n \text{ times}} = \{(x_1, x_2, \ldots x_n) : x_1, x_2, x_3 \cdots x_n \in F\}$

$(x_1, x_2 \ldots x_n) + (x_1', x_2', x_3' \cdots x_n') = (x_1 + x_1'; x_2 + x_2', \ldots x_n + x_n')$ $\quad \lambda(x_1, x_2, x_3 \cdots x_n) = (\lambda x_1, \lambda x_2, \cdots \lambda x_n)$.

5) $(F,+,\cdot)$ FIELD. $V = F[x]$ = SET OF POLYNOMIALS.

$F[x] \times F[x] \xrightarrow{+} F[x]$ Addition of polynomials.

$F \times F[x] \xrightarrow{\cdot} F[x]$ Production by a constant polynomial.

6) $(F,+,\cdot)$ FIELD. $V = M_{n\times m}(F) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}),\ a_{ij} \in F \right\}$

$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$

$\lambda(a_{ij}) = (\lambda \cdot a_{ij})$

7) Let V be an F-vector space, let $X \neq \emptyset$ set : $V \times V \xrightarrow{+} V$. $F \times V \xrightarrow{\cdot} V$.

$W = V^X = \{ f : X \to V$ functions$\}$; vectors. $f+g : X \to V : (f+g)(x) = f(x) + g(x)$.

is an F-vector space. $\lambda \cdot f : X \to V : (\lambda f)(x) = \lambda \cdot f(x)$.

8) $C^2 = \{(a+b\bar{i}, c+d\bar{i}),\ a,b,c \in R\}$ = VECTORS.

$(a+bi, c+di) + (a'+b'i, c'+d'i) = [a+a'+(b+b')i, c+c'+(d+d')i)$

$(x+yi)(a+bi, c+di) = ((x+yi)(a+bi), (x+yi)(c+di)) \Longrightarrow C^2$ is a C-vector space.

Remark: C^2 also an R-vector space $C^2 \times C^2 \xrightarrow{+} C^2$ then $R \times C^2 \longrightarrow C^2$.

SUBSPACE.

Definition.

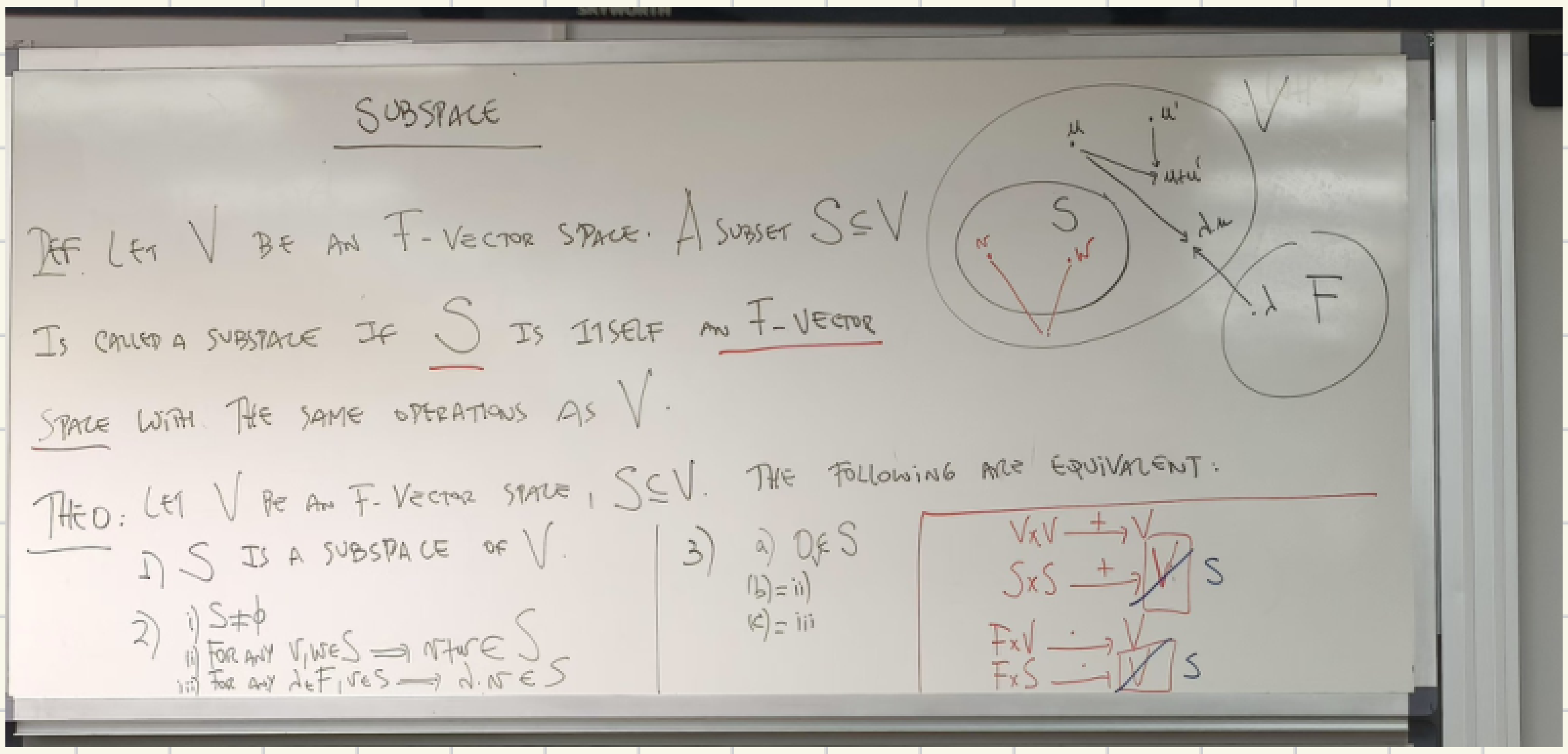
let V be an F-vector space. A subset $S \subseteq V$ is called a subspace if S is an F-vector space with the same operations as V.

THEO: let V be an F-vector space $S \subseteq V$. The following are equivalent:

1) S is a subspace of V.

2) (i) $S \neq \emptyset$ (ii) For any $v, w \in S \Rightarrow v+w \in S$. (iii) For any $\lambda \in F, v \in S \Rightarrow \lambda \cdot v \in S$.

3) (a) $0_V \in S$ (b) = (ii)   (c) = (iii).



SUBSPACE

DEF. LET V BE AN F-VECTOR SPACE. A SUBSET $S \subseteq V$ IS CALLED A SUBSPACE IF S IS ITSELF AN F-VECTOR SPACE WITH THE SAME OPERATIONS AS V.

THEO: LET V BE AN F-VECTOR SPACE, $S \subseteq V$. THE FOLLOWING ARE EQUIVALENT:

1) S IS A SUBSPACE OF V.

2) i) $S \neq \phi$

ii) FOR ANY $v, w \in S \Longrightarrow v+w \in S$

iii) FOR ANY $\lambda \in F, v \in S \longrightarrow \lambda \cdot v \in S$

3) a) $0_V \in S$

(b) = ii)

(c) = iii

$V \times V \xrightarrow{+} V$

$S \times S \xrightarrow{+} \boxed{\cancel{V}}\ S$

$F \times V \xrightarrow{\cdot} V$

$F \times S \xrightarrow{\cdot} \boxed{\cancel{V}}\ S$

3.19. CLASS 6.

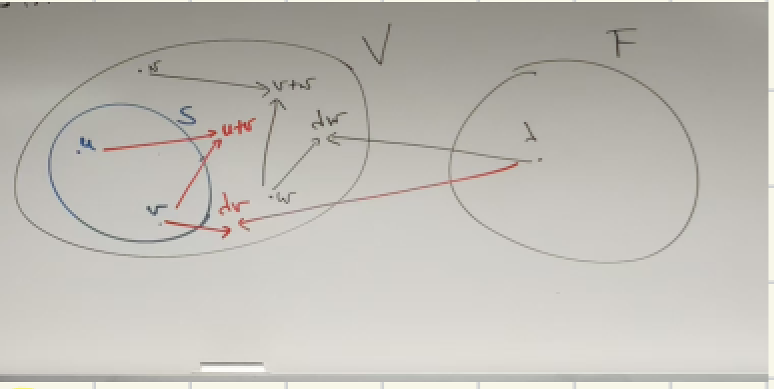
F-vector space $= (F, +, \cdot)$ field, $(V, +) + S_1 S_2 S_3 S_4$, $F \times V \xrightarrow{\cdot} V + M_1 M_2 M_3 M_4$.

SUBSPACE: $S \subseteq V$. S is an F-vector space with the same operations as V.

Question: how do we check if S is an F-vector space?

Since $V \times V \xrightarrow{+} V$ $\quad F \times V \xrightarrow{\cdot} V$. $\Rightarrow$ Restrict it into S.

$S \times S \xrightarrow{+} V$. $\quad F \times S \xrightarrow{\cdot} V$. if $S \times S \xrightarrow{+} S$. $F \times S \xrightarrow{\cdot} S$. ✓.



THEO: let V be an F-vector space, let $S \subseteq V$. The following are equal:

a) S is a subspace (= S is an F-vector space with the operations from V).

b) i) $S \neq \phi$. ii) if $v, w \in S \Rightarrow v + w \in S$. iii) if $\lambda \in F$. $v \in S \Rightarrow \lambda \cdot v \in S$.

c) i) $0 \in S$. ii) $v + w \in S$, iii) $\lambda \in F$, $v \in S$. $\lambda v \in S$.

What we need to prove:
$$\begin{tikzcd} a) \arrow[rr, Rightarrow] & & b) \arrow[dl, Rightarrow] \\ & c) \arrow[ul, Rightarrow] & \end{tikzcd}$$

Proof a) $\Rightarrow$ b).

i). We know that any vector space is not-empty. Then S is not empty.

ii. iii) If S is an F-vector space with the same operation in V. Then restrictions: $S \times S \xrightarrow{+} S$, $F \times S \xrightarrow{\cdot} S$. Then if $v, w \in S \Rightarrow v + w \in S$. $\lambda \in F$. $v \in S$, $\lambda v \in S$. ■

b) $\Rightarrow$ c).

We just need to prove i).

$S \neq \phi$. $\exists v \in S$ by iii). $0_F \in F$ $v \in S$. $\Rightarrow 0_F \cdot v = 0_V \in S$. ■

c) $\Rightarrow$ a).

By c). we know that S has addition and product. $(F, +, \cdot)$. $S \times S \xrightarrow{+} S$, $F \times S \xrightarrow{\cdot} S$.

Now we check the axiom:

S_1) $v + w + u = v + (w + u)$ $\forall v, w, u \in S$ This holds since it holds $\forall v, w, u \in V$.

S_2) we know $v + w = w + v$ $\forall v, w \in V$. So in particular, $v + w = w + v$ $\forall v, w \in S$.

S_3) = i). $0_V \in S$.

S_4) $v \in S$, we know that $-v = (-1) \cdot v$ by iii). $-1 \in F$. $v \in S \Rightarrow (-1) \cdot v \in S$.

M_1) $\lambda \cdot (v + w) = \lambda \cdot v + \lambda \cdot w$ $\forall \lambda \in F$, $v, w \in S$.

M_2) $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v$ $\forall \lambda, \mu \in F$. $v \in S$.

M_3) $(\lambda \cdot \mu) \cdot v = \lambda (\mu \cdot v)$ $\forall \lambda, \mu \in F$. $v \in S$.

M_4) $1 \cdot v = v$ $\forall v \in S$.

} This hold, since they are in V.

Example:

1) Let V be an F-vector space.

$S=\{0\}$ is a subspace $\begin{cases} 0\in S. \\ 0,0\in S \quad 0+0=0\in S. \\ \lambda\in F,\ 0\in S.\ \lambda\cdot 0=0\in S. \end{cases}$ $S=V$ is a subspace $\begin{cases} 0\in V. \\ v+w\in V. \\ \lambda\in F.\ v\in V.\ \lambda v\in V \end{cases}$

2) $V=R^3=\{(x,y,z),\ x,y,z\in R\}$. is an R-vector space.

$T=\{(x,y,1),\ x,y\in R\}\subseteq V$ is not a subspace since $0_{R^3}=(0,0,0)\notin T$.

and $(x,y,1)+(x',y',1)=(x+x',y+y',2)\notin T$.

$S=\{(x,y,0): x,y\in R\}$ is a subspace.

$(0,0,0)\in S$ ✓. $(x,y,0)+(x',y',0)=(x+x',y+y',0)\in S$ ✓.

$\lambda(x,y,0)=(\lambda x,\lambda y,0)\in S$. ✓.

3) $V=R^3$ $S=\{(x,y,z): ax+by+z=0,\ x,y,z\in R\}\subseteq R^3$.

Since $(0,0,0)\in S$. $a\cdot0+b\cdot0+c\cdot0=0$ ✓.

$(x,y,z)+(x',y',z')=(x+x',y+y',z+z')$ $a(x+x')+b(y+y')+c(z+z')=0$. ✓.

Let $\lambda\in R$. $\lambda(x,y,z)=(\lambda x,\lambda y,\lambda z)$. since $ax+by+c\cdot z=0 \Rightarrow \lambda ax+\lambda\cdot by+\lambda c\cdot z=0$. ✓.

4) $$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{mn}x_n=b_m \end{cases} \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{mn} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A \cdot x = b.$$

Let $T=\{(x_1,x_2\cdots x_n): Ax=b\}\subseteq F^n$. ← F-vector space.

$(0,0,0\cdots0)\in T \Leftrightarrow b_1=b_2=\cdots b_n=0$.

T is not a subspace if $b_1 b_2\cdots b_n\neq0$. If $(b_1\cdots b_n)=(0,0,\cdots0) \Rightarrow T=\{(x_1,x_2\cdots x_n): A\cdot x=0\}$ is a subspace.

$(0,0,0\cdots0)\in T$.

$x=(x_1,x_2\cdots x_n)$, $x'=(x_1',x_2',x_3'\cdots x_n')\in T \Rightarrow Ax=0.\ Ax'=0. \Rightarrow A(x+x')=Ax+Ax'=0. \Rightarrow x+x'\in T.$

$x=(x_1,x_2,\cdots x_n)\in T,\ \lambda\in F. \Rightarrow A\cdot(\lambda x)=A(\lambda\cdot Id)\cdot x=\lambda(Ax)=(\lambda\cdot Id)\cdot 0=0. \Rightarrow \lambda\cdot x\in T.$

5) C is a C-vector space, $R\subseteq C$. R is not a subspace of C vector space.

i) $0\in R$, ii) $x,y\in R. \Rightarrow (x+0i)+(y+0i)=x+y=(x+y)+0i\in C$. ✓.

iii) $\lambda\in C$. $x\in R. \Rightarrow \lambda x\in R. \Rightarrow$ Not true since if $\lambda=i$. ✗

In otherwise. C is an R-vectorspace.

$R\subseteq C$, R is a subspace.

i) $0=0+0i\in R$. ii) $x,y\in R, \Rightarrow x+y=x+0i+y+0i=(x+y)+0i\in R$.

iii) $\lambda\in R,\ x\in R. \Rightarrow \lambda\cdot x=\lambda(x+0i)=\lambda x+0i\in R$.

→ R-vector space $\Rightarrow R^R$ is an R-vector space.

6) $R^R=\{f: R\to \underline{R}\ \text{functions}\}$.

We have already checked $S_1S_2S_3S_4M_1M_2M_3M_4$ ✓.

Let $C(R)=\{f:R\to R\ \text{continuous}\}$ is a subspace of R^R.

let a function $O(x)$. $O: R \to R$, $O(x)=0$ is continuous. then $O(x) \in C(R)$

let $f,g \in C(R) \Rightarrow f+g \in C(R)$. $\lambda \in R$. $f \in C(R) \Rightarrow \lambda f \in C(R)$.

$\lim(f+g) = \lim f + \lim g$. $\lim(\lambda f) = \lambda \cdot \lim f$.

THEO: If S_1, S_2 are subspace of a F-vector space of V. $\Rightarrow S_1 \cap S_2$ is a subspace of V.

2). If $\{S_i, i \in T\}$ ... ... $\Rightarrow \bigcap_{i \in I} S_i$ ... ...

Proof:

1). $0 \in S_1$, $0 \in S_2$. $\Rightarrow 0 \in S_1 \cap S_2$ ✓.

If $v,w \in S_1 \cap S_2. \Rightarrow \begin{cases} v,w \in S_1 \Rightarrow v+w \in S_1 \\ v,w \in S_2 \Rightarrow v+w \in S_2 \end{cases} \Rightarrow v+w \in S_1 \cap S_2.$

If $\lambda \in F$, $v \in S_1 \cap S_2 \Rightarrow \begin{cases} \lambda \in F, v \in S_1 \Rightarrow \lambda \cdot v \in S_1 \\ \lambda \in F, v \in S_2 \Rightarrow \lambda \cdot v \in S_2. \end{cases} \Rightarrow \lambda \cdot v \in S_1 \cap S_2.$

Example:

$S_1 = \{(x,y,0) : x,y \in R\} \subseteq R^3$. $S_2 = \{(x,0,z), x,z \in R\} \subseteq R^3$ both are subspace.

$S_1 \cap S_2 = \{(x,0,0) : x \in R\}$ subspace ✓.

$S_1 \cup S_2 = \{(x,y,0) : x,y \in R\} \cup \{(x,0,z) : x,z \in R\}$ is NOT a subspace.

Since $(1,1,0) + (1,0,1) = (2,1,1) \notin S_1 \cup S_2$.

Def:

1) let $S_1.S_2$ be subspace of an F-vector space V. Then S_1+S_2 is the SMALLEST subspace of V. containing $S_1 \cup S_2$. with respect to inclu.

Proof: S_1+S_2 is a subspace containing $S_1 \cup S_2$, $S_1+S_2 \subseteq R^3$.

$(x,y,0) \in S_1 \subseteq S_1+S_2 \subseteq R^3$. $(0,0,z) \in S_2 \subseteq S_1+S_2 \subseteq R^3$.

$(x,y,0)+(0,0,z) \in S_1+S_2$, S_1+S_2 is a subspace $\Rightarrow (x,y,z) = (x,y,0)+(0,0,z) \in S_1+S_2$.

$\Rightarrow R^3 \subseteq S_1+S_2 \subseteq R^3. \Rightarrow S_1+S_2 = R^3.$

Def:

2) let $S_1, S_2 \cdots S_n$ be subspace of V. Then $S_1+S_2+\cdots+S_n$ is the smallest subspace of V. Containing $S_1 \cup S_2 \cup S_3 \cdots \cup S_n$.

THEO:

1) $S_1+S_2 = \{v_1+v_2, v_1 \in S_1, v_2 \in S_2.\}$

2) $S_1+S_2+\cdots+S_n = \{v_1+v_2+\cdots+v_n, v_1 \in S_1, v_2 \in S_2 \cdots v_n \in S_n\}$.

PROOF: WE HAVE TO PROVE THAT S_1+S_2 IS THE $\overset{(c)}{\underline{\text{SMALLEST}}}$ $\overset{(a)}{\underline{\text{SUBSPACE}}}$. $\overset{(b).}{\underline{\text{CONTAINING } S_1 \text{ and } S_2}}$.

(a). i) $0 = \overset{\in S_1}{0} + \overset{\in S_2}{0}$. $\in S_1+S_2$. since S_1, S_2 are subspace.

ii) v_1+v_2, w_1+w_2, $v_1,w_1 \in S_1$. $v_2,w_2 \in S_2 \Rightarrow (v_1+v_2)+(w_1+w_2) = \overset{\in S_1}{(v_1+w_1)} + \overset{\in S_2}{(v_2+w_2)} \in S_1+S_2$.

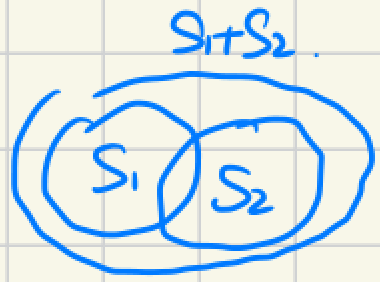
iii) $\lambda \in F$. v_1+v_2 $v_1 \in S_1$ $v_2 \in S_2 \Rightarrow \lambda(v_1+v_2) = \lambda v_1 + \lambda v_2 \in S_1+S_2$.

(b) $S_1 \subseteq S_1+S_2$ since $v \in S_1 \Rightarrow v = \overset{\in S_1}{v} + \overset{\in S_2}{0} \in S_1+S_2$.

$S_2 \subseteq S_1+S_2$ since $v \in S_2 \Rightarrow v = \overset{\in S_2}{v} + \overset{\in S_1}{0} \in S_1+S_2$.

(c) We have to prove that S_1+S_2 is the SMALLEST one. let U be a subspace containing $S_1 \cup S_2$.
We have to prove that $S_1+S_2 \subseteq U$.

$v_1+v_2 \in S_1+S_2 \Rightarrow \begin{cases} v_1 \in S_1 \subseteq U. \\ v_2 \in S_2 \subseteq U. \end{cases} \Leftrightarrow v_1, v_2 \in U$. U is a subspace, $\Rightarrow v_1+v_2 \in U$.



Remark: S_1+S_2 is unique.

U_1 is the smallest subspace containing $S_1 \cup S_2$. $\Rightarrow U_1 \subseteq U_2$, $U_2 \subseteq U_1$.
U_2 is the smallest subspace containing $S_1 \cup S_2$

Example:

$R^3 = S_1+S_2 = \{(x,y,0) : x,y \in R\} + \{(x,0,z), x,z \in R\}$.
$(x,y,z) = (x,y,0) + (0,0,z) \in S_1+S_2$.   Not unique.
$= (0,y,0) + (x,0,z) \in S_1+S_2$.

$R^3 = S_1+S_2 = \{(x,y,0) : x,y \in R\} + \{(0,0,z), z \in R\}$.
$(x,y,z) = (a,b,0) + (0,0,c) \Leftrightarrow (x,y,z) = (a,b,c)$
$= (x,y,0) + (0,0,z)$ In a unique way.

3.24. CLASS 7

Def: the sum of two subspaces S_1, S_2 of an F-vector space V is the $\overset{(iii)}{\text{SMALLEST}}$ $\overset{(i)}{\text{SUBSPACE}}$ of V that contains $S_1 \cup S_2$. (ii)

THEO: $S_1+S_2 = \{v_1+v_2, v_1 \in S_1, v_2 \in S_2\}$.

Proof: Let $X = \{v_1+v_2, v_1 \in S_1, v_2 \in S_2\}$. We want to prove that $X = S_1+S_2$.
Then we have to check.
i) X is a subspace of V.
ii) $S_1 \cup S_2 \subseteq X$.
iii) If U is a subspace of V. $S_1 \cup S_2 \subseteq U \Rightarrow X \subseteq U$

$\Bigg\} \Rightarrow X = S_1+S_2$.

Example:

$R^3 = \{(x,y,0), x,y \in R\} + \{(0,y,z) : y,z \in R\} \overset{\text{THEO}}{=} \{(x,y,0)+(0,y',z'), x,y,y',z' \in R\} \subseteq R^3$.

For any $(x,y,z) \in R^3$, we have $(x,y,z) = (x,y,0) + (0,0,z) \in S_1+S_2$.
$= (x,0,0) + (0,y,z) \in S_1+S_2$.

$\Rightarrow$ The way in which we write (x,y,z) as v_1+v_2, $v_1 \in S_1$. $v_2 \in S_2$ is not unique.

$$R^3 = \underset{S_1}{\{(x,y,0), x,y \in R\}} \oplus \underset{S_2.}{\{(0,0,z) : z \in R\}}$$

$(x,y,z) = (x,y,0) + (0,0,z)$
$= (a,b,0) + (0,0,c) \Leftrightarrow \begin{cases} a = x \\ b = y \\ c = z. \end{cases}$

Def: 1) A sum of two subsequences, S_1+S_2, is called DIRECT SUM if any vector in S_1+S_2 can be written in a unique way as $v_1+v_2 \in S_1+S_2$ $v_1 \in S_1$, $v_2 \in S_2$.
That is, $w = v_1+v_2 = v_1'+v_2'$. $v_1, v_1' \in S_1$ $v_2, v_2' \in S_2$ $\Rightarrow \begin{cases} v_1 = v_1' \\ v_2 = v_2'. \end{cases}$

2). A sum of n subspace $S_1+S_2+\cdots S_n$ is DIRECT SUM if $w = v_1+v_2+v_3+\cdots+v_n = v_1'+v_2'+v_3'+\cdots+v_n' \in S_1+S_2+\cdots+S_n$.
for $v_1, v_1' \in S_1$, $v_2, v_2' \in S_2 \cdots v_n, v_n' \in S_n$.

$\Rightarrow v_1 = v_1' \quad v_2 = v_2' \cdots \quad v_n = v_n'$

$\Rightarrow S_1 \oplus S_2 \oplus S_3 \cdots \oplus S_n$.

Example:

$\mathbb{R}^3 = S_1 + S_2 \Rightarrow$ is not direct.

$= S_1 \oplus S_3 = \{(x,y,0), x,y \in \mathbb{R}\} \oplus \{(0,0,z), z \in \mathbb{R}\}$. ①

$= T_1 \oplus T_2 \oplus S_3 = \{(x,0,0), x \in \mathbb{R}\} \oplus \{(0,y,0), y \in \mathbb{R}\} \oplus \{(0,0,z), z \in \mathbb{R}\}$. ②

If we have any (x,y,z). let's do ① way:

$(x,y,z) = (a,b,0) + (0,0,c)$. $\Rightarrow a = x,\ b = y,\ c = z$. then for $\forall (x,y,z)$ we have only one solution.

② way:

$(x,y,z) = (a,0,0) + (0,b,0) + (0,0,c) \Rightarrow a = x.\ b = y,\ c = z$ " . . . ".

Now, let's assume $(x,y,z) = (a,0,0) + (0,b,b) + (c,0,c)$. $\Leftrightarrow \begin{cases} x = a+c \\ y = b \\ z = b+c \end{cases} \Leftrightarrow \begin{cases} b = y \\ c = z-y. \\ a = x-z+y. \end{cases}$

PROBLEMS:

1) How do we know that $U = S_1 + S_2$?

If $S_1 + S_2 \subseteq U$ and any $w \in U$ we can be written as $v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2 \Rightarrow w \in S_1 + S_2$

2) How do we know that $U = S_1 \oplus S_2$?

For any $v \in U$ check that if it can be written as $v_1 + v_2 \in S_1 + S_2$ st $v_1 \in S_1$, $v_2 \in S_2$ in a unique way.

We have a theo.

THEOREM:

1) $S_1 \oplus S_2$ is a direct sum $\Leftrightarrow 0 = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2 \Rightarrow v_1 = 0, v_2 = 0$.

2) $S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is a direct sum $\Leftrightarrow 0 = v_1 + v_2 + \cdots + v_n$, $v_i \in S_i$, $i = 1,2 \ldots n$. $\Rightarrow v_1 = v_2 = v_3 = \cdots = v_n = 0$.

Proof:

1) $\Rightarrow$) We know that any $w \in S_1 + S_2$ can be written in a unique way as $w = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2$. In particular:

We know that $0 = \overset{\in S_1}{0} + \overset{\in S_2}{0}$ in a unique way.

Then $0 = \overset{\in S_1}{v_1} + \overset{\in S_2}{v_2} \Rightarrow v_1 = 0$. $v_2 = 0$.

$\Leftarrow$) Assume $w = v_1 + v_2 = v_1' + v_2'$ $v_1, v_1' \in S_1$, $v_2, v_2' \in S_2$. We want to prove that $v_1 = v_1'$ $v_2 = v_2'$.

Since $0 = w - w = v_1 + v_2 - (v_1' + v_2') = \overset{\in S_1}{v_1 - v_1'} + \overset{\in S_2}{v_2 - v_2'}$ By hypothesis $v_1 - v_1' = 0$ $v_2 - v_2' = 0$. $\Rightarrow v_1 = v_1'$ $v_2 = v_2'$. ∎

Corollary: $S_1 + S_2 = S_1 \oplus S_2$ is a direct sum $\Leftrightarrow S_1 \cap S_2 = \{0\}$.

Proof: We know that $S_1 + S_2 = S_1 \oplus S_2 \Leftrightarrow 0 = v_1 + v_2$, $v_1 \in S_1$ $v_2 \in S_2 \Rightarrow v_1 = 0, v_2 = 0$.

let's see that $(0 = \overset{\in S_1}{v_1} + \overset{\in S_2}{v_2} \Rightarrow v_1 = 0 = v_2) \Leftrightarrow S_1 \cap S_2 = \{0\}$.

$\Rightarrow$) $v \in S_1 \cap S_2 \Rightarrow -v \in S_1 \cap S_2$. Since $S_1 \cap S_2$ is a subspace and $-v = (-1) \cdot v$.

$0 = \overset{\in S_1 \cap S_2}{v} + \overset{\in S_1 \cap S_2}{(-v)} \Rightarrow \begin{cases} v = 0 \\ -v = 0 \end{cases} \Rightarrow \{0\} \subseteq S_1 \cap S_2 \subseteq \{0\} \Rightarrow S_1 \cap S_2 = \{0\}$ (we know that $\{0\} \subseteq S_1 \cap S_2$ trivially).

$\Leftarrow$) let $0 = v_1 + v_2$ $v_1 \in S_1$ $v_2 \in S_2$. $\Rightarrow \overset{\in S_1}{v_1} = \overset{\in S_2}{-v_2}$. $\in S_1 \cap S_2 = \{0\}$.

$\Rightarrow \begin{cases} v_1 = 0 \\ -v_2 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = 0 \\ v_2 = 0 \end{cases} \Rightarrow S_1 \oplus S_2$. ∎

$0 = v_1 + v_2$. for $v_1 \in S_1$. $v_2 \in S_2$.

$v_1 = -v_2$. $\Rightarrow v_1 \in S_2. \Rightarrow v_1 \in S_1 \cap S_2$.

$v_2 \in S_1 \cap S_2$. $\Rightarrow v_1 = v_2 = 0$. $\Rightarrow S_1 \oplus S_2$.

Example: let $S_1 = \{(x,0,x), x \in \mathbb{R}\}$. $S_2 = \{(x,0,-x), x \in \mathbb{R}\}$.

$S_1 \oplus S_2 = \{(a,0,a) + (b,0,-b),\ a,b \in \mathbb{R}\} = \{(a+b, 0, a-b),\ a,b \in \mathbb{R}\}$.

let's take any $(x,0,y)$ for $x,y \in \mathbb{R}$. we have: $x = a+b$. $y = a-b$. $\Leftrightarrow \begin{cases} a = \frac{x+y}{2} \\ b = \frac{x-y}{2} \end{cases}$.

Question = What is $S_1 \cap S_2$?

$(x,y,z) \in S_1 \cap S_2 \Leftrightarrow \begin{matrix} (x,y,z) \in S_1 \\ (x,y,z) \in S_2 \end{matrix} \Leftrightarrow \begin{cases} y=0 \text{ and } x=z \\ y=0 \text{ and } x=-z \end{cases} \Leftrightarrow x=y=z=0 \Rightarrow S_1 \cap S_2 = \{(0,0,0)\}$.

Remark: The corollary is not true for $n \geq 3$

$S_1 + S_2 + S_3 + \cdots S_n = S_1 \oplus S_2 \oplus S_3 \cdots \oplus S_n \not\Leftrightarrow S_1 \cap S_2 = \{0\},\ S_2 \cap S_3 = \{0\} \cdots S_{n-1} \cap S_n = \{0\}$.

BUT $S_1 + S_2 + S_3 = S_1 \oplus S_2 \oplus S_3 \Longleftrightarrow S_1 \cap (S_2 + S_3) = \{0\}\quad S_2 \cap (S_1 + S_3) = \{0\}$.

We will see in Algebra B.

LINEAR COMBINATION.

definition:

let V be an F-vector space $v_1, v_2 \cdots v_n, w \in V$.

We say that w is a Linear Combination (LC) of $v_1, v_2 \cdots v_n$ if $\exists \lambda_1, \lambda_2 \ldots \lambda_n \in F$ st

$w = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n \cdot v_n$.

Example:

1) Check that $(1,0,1)$ is a L.C of $(2,1,3)$ $(1,0,0)$ and $(0,1,1)$.

$(1,0,1) = \lambda_1(2,1,3) + \lambda_2(1,0,0) + \lambda_3(0,1,1) \Leftrightarrow \begin{cases} 1 = 2\lambda_1 + \lambda_2 \\ 0 = \lambda_1 + \lambda_3 \\ 1 = 3\lambda_1 + \lambda_3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = -\lambda_3 \\ 3\lambda_1 - \lambda_1 = 1 \\ \lambda_2 = 2\lambda_1 - 1 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = -\lambda_3 \\ \lambda_1 = \frac{1}{2} \\ \lambda_2 = 0 \end{cases}$

$= \frac{1}{2}(2,1,3) + (-\frac{1}{2})(0,1,1)$.

Def: $S_1 + S_2 + \cdots + S_n$ = Sum of subspaces = SMALLEST SUBSPACE of V. contain $S_1 \cup S_2 \cdots \cup S_n$

Def: $S_1 \oplus S_2$ = Direct sum of " ".

Def: L.C. = sum of scalars times vectors.

Definition: let V be an F-vector space, $v_1 \cdots v_n \in V$.

$\mathrm{SPAN}(v_1, v_2 \ldots v_n) = \mathrm{SPAN}(\{v_1, v_2 \ldots v_n\})$. is the smallest (with respect to function). subspace of V containing the set $\{v_1, v_2, \cdots v_n\}$.

Remark:

let $U = \mathrm{span}(v_1, v_2 \cdots v_n)$. U is a subspace and $v_i \in U$ for any $i = 1, 2, \cdots n$.

$v_i \in U \Leftrightarrow S_i = \{\lambda \cdot v_i,\ \lambda \in F\} \subseteq U$.

$\Rightarrow$) $v_i \in U \Rightarrow \lambda \cdot v_i \in U$. since U is a subspace.

$\Leftarrow$) $v_i = 1 \cdot v_i \in U$.

THEO: $\mathrm{Span}(v_1, v_2 \cdots v_n)$ = smallest subspace of U containing $S_1 \cup S_2 \cup S_3 \cdots \cup S_n$. $S_i = \{\lambda \cdot v_i, \lambda \in F\}$.

$= S_1 + S_2 + \cdots + S_n$.

$= \{\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3, \cdots \lambda_n v_n,\ \lambda_i \in F\}$ = set of L.C. of $v_1, v_2 \cdots v_n$.

Definition.

We say that the set $\{v_1, v_2, \cdots v_n\}$ spans the F-vector space V if

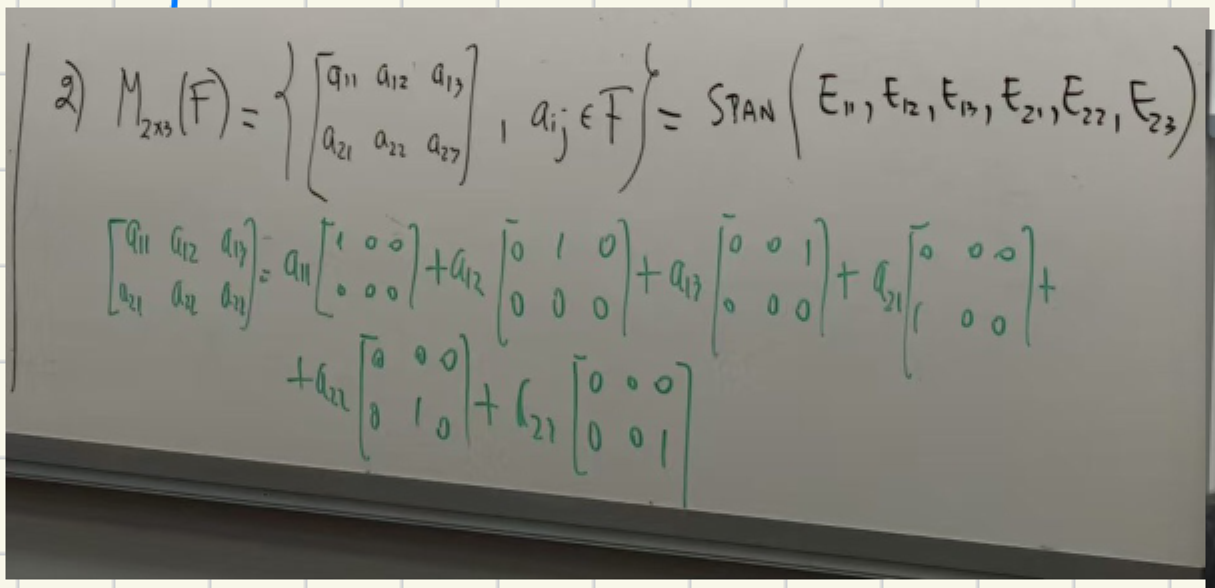
$V = \mathrm{span}(v_1, v_2 \cdots v_n)$ = set of L.C. of $v_1, v_2 \ldots v_n = \{\lambda v_1, \lambda \in F\} + \{\lambda v_2, \lambda \in F\} \cdots + \{\lambda v_n, \lambda \in F\}$.

Example:

1) $F^2 = \{(x,y) : x,y \in F\} = \text{span}(\{(1,0),(0,1)\})$. $(x,y) = x\cdot(1,0) + y\cdot(0,1)$.

$F^n = \text{Span}(\{P_1, P_2, \cdots P_n\})$ $P_1 = (1,0,0,\cdots 0)$. $P_2 = (0,1,0,0\cdots 0)$ $\cdots$ $P_n = (0,0,\cdots 0,1)$.

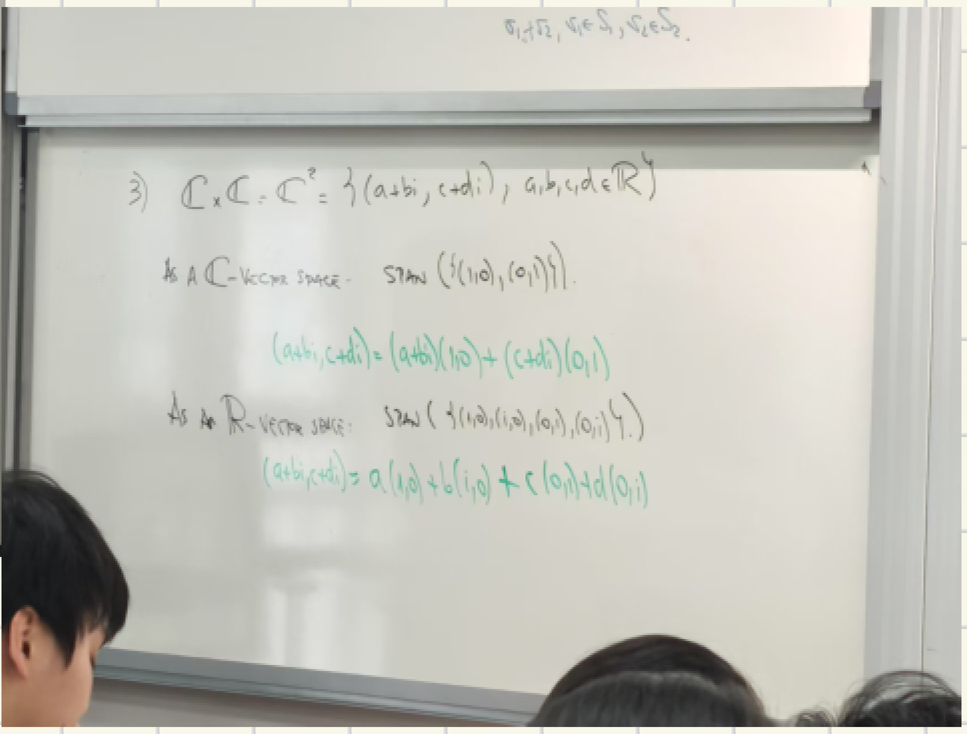
2)



2) $M_{2\times3}(F) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, a_{ij} \in F \right\} = \text{SPAN}(E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$

$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_{21}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} +$

$+ a_{22}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_{23}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



v_1+v_2, $v_1 \in S_1$, $v_2 \in S_2$.

3) $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2 = \{(a+bi, c+di), a,b,c,d \in \mathbb{R}\}$

AS A $\mathbb{C}$-VECTOR SPACE: SPAN $(\{(1,0),(0,1)\})$.

$(a+bi, c+di) = (a+bi)(1,0) + (c+di)(0,1)$

AS AN $\mathbb{R}$-VECTOR SPACE: SPAN $(\{(1,0),(i,0),(0,1),(0,i)\}.)$

$(a+bi, c+di) = a(1,0) + b(i,0) + c(0,1) + d(0,i)$

3.26. CLASS 8.

Vector space:
- $(F, +, \cdot)$ field.
- $(V, +)$
- $F \times V \to V$.

} + Axioms.

F-subspaces = subsets of F-vector space which are F-vector space.

Intersetion of subspaces is a subspace.

Union is not always a subspace.

Sum on subspace: $S_1 + S_2 + \cdots + S_n \overset{\text{Def}}{=}$ SMALLEST SUBSPACE OF V CONTAINING $S_1 \cup S_2 \cup S_3 \cdots \cup S_n$.

$\overset{\text{THEO}}{=} \{v_1 + v_2 + \cdots + v_n, v_i \in S_i, i = 1,2,\cdots n\}$.

In particular: $S_i = \{\lambda v_i, \lambda \in F\}$ (Check that S_i is a subspace of V)

$S_1 + S_2 + \cdots + S_n = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \lambda_1 \cdots \lambda_n \in F\}$.

$\overset{\text{Def}}{=}$ set of LC of $v_1 \cdots v_n$.

$\overset{\text{Def}}{=}$ SMALLEST SUBSET OF V CONTAINING $S_1 \cup S_2 \cup \cdots \cup S_n$.

Proposition " " " " $\{v_1 \ldots v_n\}$.

$\overset{\text{Def}}{=}$ SPAN $(v_1 \ldots v_n)$.

Proposition: U is a vector space: $v \in U \Longleftrightarrow \{\lambda v_i, \lambda \in F\} \subseteq U$.

Example:

1) C-vector space $C^2 \Rightarrow C^2 = \text{SPAN}((1,0),(0,1))$.

$(a+bi, c+di) = (a+bi)\cdot(1,0) + (c+di)\cdot(0,1)$.

$\{(1,0),(0,1)\}$ does not span the R-vector space C^2: $(i, 1+i) \neq u_1(1,0) + u_2(0,1)$, $u_1, u_2 \in R$.

2) R-vector space $C^2 \Rightarrow C^2 = \text{span}\{(1,0),(i,0),(0,1),(0,i)\}$.

$(a+bi, c+di) = a(1,0) + b(i,0) + c(0,1) + d(0,i)$. $a,b,c,d \in R$.

3) Let $S = \{(x+y, x, 2y) : x,y \in R\} \subseteq R^3$. Find a spanning set of S.

$(x+y, x, 2y) = (x,x,0) + (y,0,2y) = x(1,1,0) + y(1,0,2) \Rightarrow S = \text{span}\{(1,1,0),(1,0,2)\}$

4). $\mathrm{Span}((1,2,0),(0,1,1)) \overset{\text{DEF}}{=}$ SMALLEST SUBSPACE OF R^3 CONTAINING $\{(1,2,0),(0,1,1)\}$.

$= \{a(1,2,0)+b(0,1,1) : a,b\in R\}$.

$= \{(a, 2a+b, b) \cdot a,b\in R\} \not\ni (2,3,10)$.

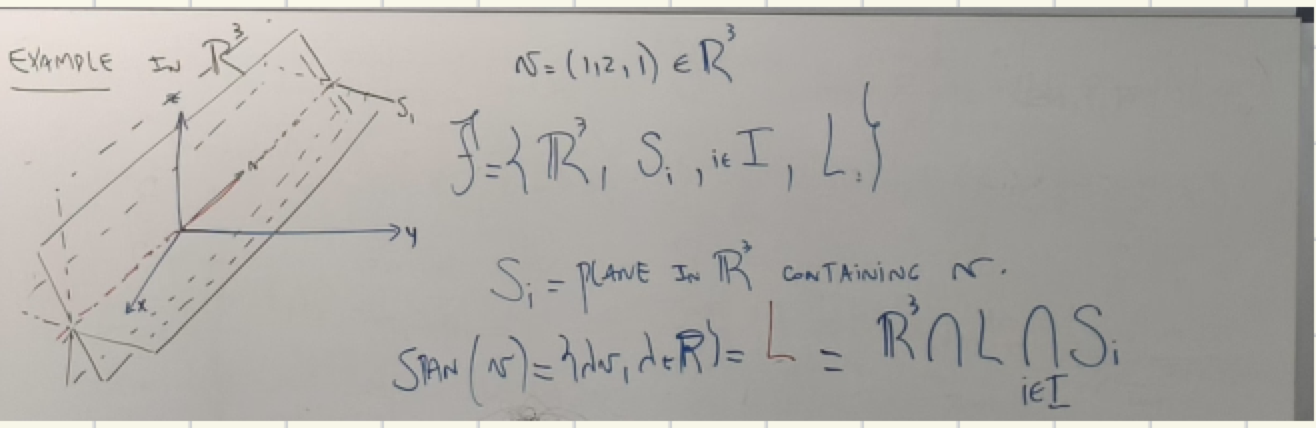
$= \{(x,y,z) : y = 2x+z,\ x,z\in R\}. = \{(x, y, y-2x),\ y, x\in R\}$.

$= \mathrm{span}((1,0,2),(0,1,1))$.

THEO: let V be an F-vector space, $X\subseteq V$ a subset.

Consider: $\mathcal{f} = \{S\subseteq V,\ S \text{ is a subspace of } V,\ X\subseteq S\}$ Then $\mathrm{span}(X) = \bigcap_{S\in\mathcal{f}} S$.

Proof: $\mathcal{f}\neq\phi$ since $V\in\mathcal{f}$ $\mathrm{span}(x) = \bigcap_{S\in\mathcal{f}} S \Leftrightarrow \begin{cases}\bigcap_{S\in\mathcal{f}} S \text{ is a subspace of } V. \\ x\subseteq \bigcap_{S\in\mathcal{f}} S. \text{ since } x\subseteq S\ \forall S\in\mathcal{f} \\ \text{SMALLEST. } \bigcap_{S\in\mathcal{f}} S \subseteq S\ \forall S\in\mathcal{f}.\end{cases}$



$v = (1,2,1)\in\mathbb{R}^3$

$\mathcal{f} = \{\mathbb{R}^3, S_i, i\in I, L\}$

S_i = PLANE IN $\mathbb{R}^3$ CONTAINING v.

$\mathrm{SPAN}(v) = \{\lambda v, \lambda\in\mathbb{R}\} = L = \mathbb{R}^3\cap L \bigcap_{i\in I} S_i$

Remark:

1) $S_1+S_2+\cdots+S_n = \{v_1+v_2+\cdots+v_n,\ v_i\in S_i\} = \{\lambda_1 v_1+\lambda_2 v_2+\cdots+\lambda_n v_n,\ v_i\in S_i\}$.

$= \mathrm{Span}(x)$. $X = S_1\cup S_2\cup\ldots\cup S_n$.

We can apply the theorem for $X = S_1\cup S_2\cup\cdots\cup S_n$.

2) $\mathrm{SPAN}\{v_1\ldots v_n\}$, $X = \{v_1, v_2, v_3\cdots v_n\}$.

We can apply the theorem for this X.

FINITE DIMENSIONAL VECTOR SPACES

Def: An F-vector space V is finite dimensional if $\exists X\subseteq V$. X finite, st $V = \mathrm{span}(X)$.

That is $X = \{v_1\ldots v_n\}$, $V = \mathrm{span}(v_1 v_2\ldots v_n) = \{\lambda_1 v_1+\lambda_2 v_2+\cdots+\lambda_n v_n,\ \lambda_1,\lambda_2\cdots\lambda_n\in F\}$.

V is Infinite dimensional if it is not finite dimensional.

Notation: V has finite dimension $\dim_F V < \infty$.

V has infinite dimension $\dim_F V = \infty$.

Example:

1) $\dim_F F^n < \infty$ since $(x_1 x_2\ldots x_n) = x_1(\overset{e_1}{1,0,\cdots 0}) + x_2(\overset{e_2}{0,1,0,0\cdots 0}) + \cdots + x_n(\overset{e_n}{0,0,\cdots 0,1})$.

2) $\dim_F M_{m\times n}(F) < \infty$ $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn}\end{bmatrix} = a_{11}E_{11} + a_{12}E_{12} + \cdots + a_{mn}E_{mn}$. $(E_{ij})_{ks} = \begin{cases} 1 & (i,j)=(k,s) \\ 0 & (k,s)\neq(i,j).\end{cases}$

$M_{m\times n}(F) = \mathrm{span}(E_{11}, E_{12}, \cdots E_{1n}, \ldots E_{m1}\cdots E_{mn})$.

3) $S = \mathrm{span}((1,2,0),(0,1,1)) = \{(x, 2x+z, z),\ x,z\in R\}$, $\dim_R S < \infty$.

4) $\dim_F F[x] = \infty$

Assume $\dim_F F[x] < \infty$, $\exists X = \{p_1(x), p_2(x)\cdots p_s(x)\}\subseteq F[x]$ st $F[x] = \mathrm{span}(X) = \mathrm{span}(p_1(x), p_2(x),\cdots p_s(x)) = \{\lambda_1 p_1(x)+\cdots+\lambda_s p_s(x), \lambda_i\in F$

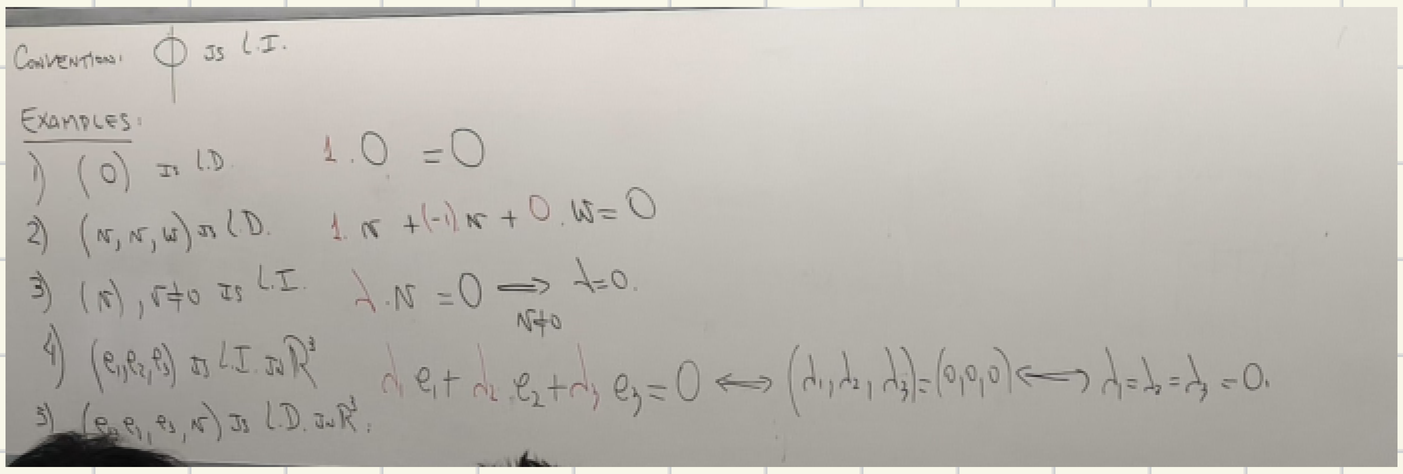
If $m = \max\{\deg p_i(x), p_i(x), i=1,2,\ldots s\}$. $\Rightarrow x^{m+1}\notin \mathrm{span}(p_1 p_2\cdots p_s) = F[x]$. Contradiction.

Def: A sequence of vectors $(v_1 v_2 \cdots v_n)$ in an F-vector space, V is called Linearly Dependent. LD.
If $\exists \lambda_1, \lambda_2 \cdots \lambda_n \in F$, $(\lambda_1, \lambda_2 \cdots \lambda_n) \neq (0,0,0,\cdots 0)$ st $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n = 0$.

Def: A sequence is Linearly Independent LI. If is not LD.

An Infinite sequence of vectors is called LD if it contains a finite subsequence is LD

LI is in the same property.



CONVENTION: ϕ IS L.I.

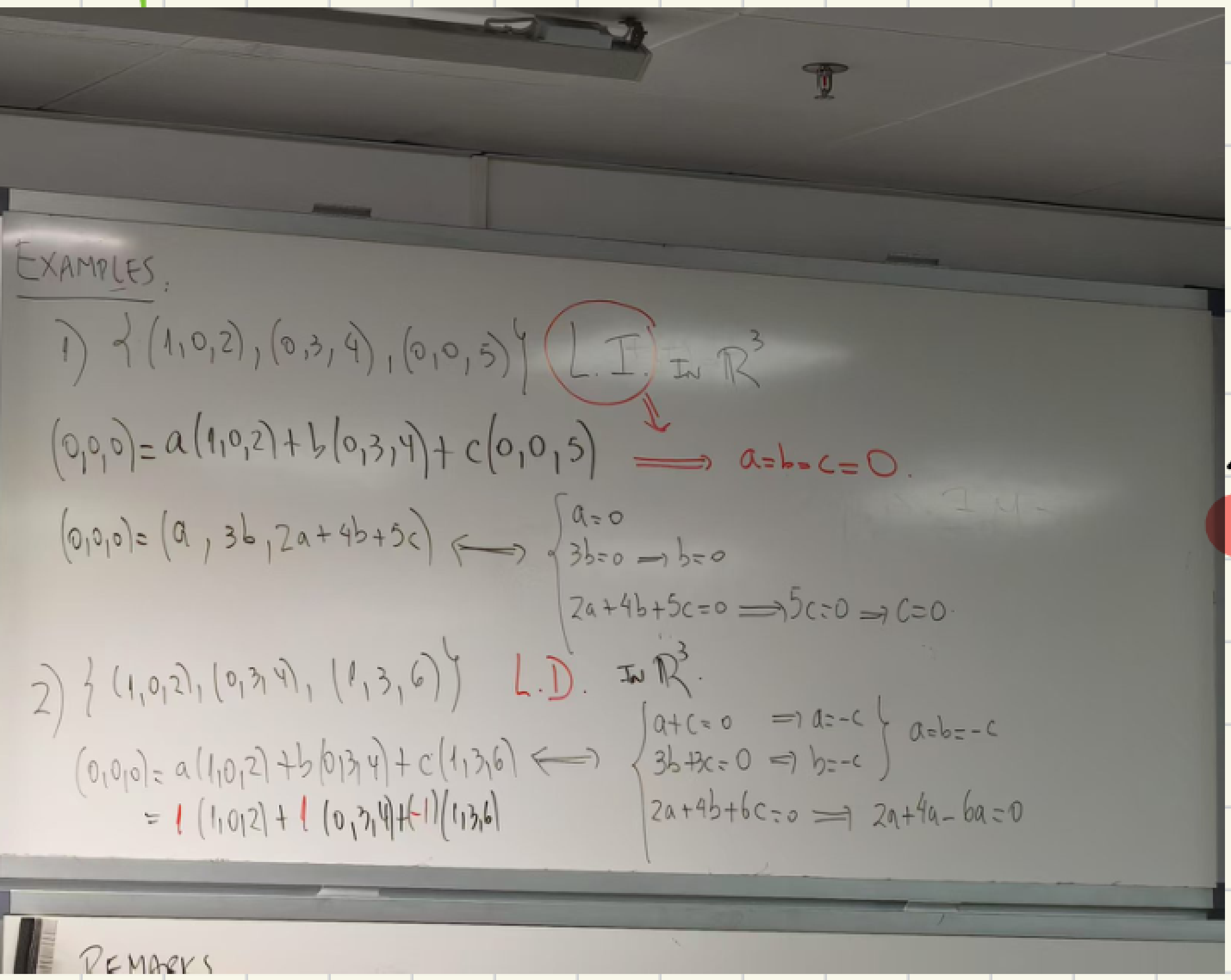
EXAMPLES:

1) (0) IS L.D. $1.0 = 0$
2) (v, v, w) IS L.D. $1.v + (-1)v + 0.w = 0$
3) $(v), v \neq 0$ IS L.I. $\lambda.v = 0 \underset{v \neq 0}{\Longrightarrow} \lambda = 0.$
4) (e_1, e_2, e_3) IS L.I. IN $\mathbb{R}^3$ $\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 = 0 \iff (\lambda_1, \lambda_2, \lambda_3) = (0,0,0) \iff \lambda_1 = \lambda_2 = \lambda_3 = 0.$
5) (e_1, e_2, e_3, v) IS L.D. IN $\mathbb{R}^3$.

3.31. CLASS 9.

Remark: We always write 0 as a Trival LC of $v_1, v_2 \dots v_n$ $0 = 0 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_n$.
$(v_1, v_2, \dots v_n)$ is L.D $\Rightarrow$ 0 can be written as a L.C. of $v_1 v_2 \dots v_n$ in a non-trival way.

Example.



EXAMPLES:

1) $\{(1,0,2), (0,3,4), (0,0,5)\}$ L.I. IN $\mathbb{R}^3$

$(0,0,0) = a(1,0,2) + b(0,3,4) + c(0,0,5) \Longrightarrow a = b = c = 0.$

$$(0,0,0) = (a, 3b, 2a+4b+5c) \iff \begin{cases} a = 0 \\ 3b = 0 \to b = 0 \\ 2a+4b+5c = 0 \Longrightarrow 5c = 0 \Rightarrow c = 0. \end{cases}$$

2) $\{(1,0,2), (0,3,4), (1,3,6)\}$ L.D. IN $\mathbb{R}^3$.

$$(0,0,0) = a(1,0,2) + b(0,3,4) + c(1,3,6) \iff \begin{cases} a+c = 0 \Rightarrow a = -c \\ 3b+3c = 0 \Rightarrow b = -c \\ 2a+4b+6c = 0 \Longrightarrow 2a+4a-6a = 0 \end{cases} \quad a = b = -c$$

$= 1(1,0,2) + 1(0,3,4) + (-1)(1,3,6)$

REMARKS

Remark:

1) ϕ is LI.
2) $\{v, v \in V, v \neq 0\}$ L.I.
3) $\{v, v, v_1, v_2 \dots v_n\}$ LD. since $v + (-1) \cdot v + 0 \cdot v_1 + 0 v_2 + \dots 0 v_n = 0$.
4) $(0, v_2, v_3, \dots)$ LD. $0 = 1 \cdot 0 + 0 \cdot v_2 \dots$
5) (v, w) LI $\iff v \neq w \neq 0$. and $w \neq \lambda \cdot v, \lambda \in F$.
$\iff v, w \neq 0, w \notin span(v)$

let's prove (5).

$\Leftarrow$) By contradiction: assume (v, w) are LD. $\Rightarrow \exists a, b$ st $0 = a \cdot v + b w$
if $b = 0 \Rightarrow 0 = a \cdot v$, $v \neq 0 \Rightarrow a = 0 \Rightarrow (a, b) = (0, 0)$ Contradiction
if $b \neq 0. \Rightarrow w = -\frac{a}{b} v = \lambda \cdot v$, $\lambda = -\frac{a}{b}$ Contradiction since $w \neq \lambda \cdot v$. $w \notin span(v)$.

$\Rightarrow$) Assume (v, w) L.I. If $v = 0 \Rightarrow (v, w) = (0, w)$ LD. If $w = 0$ is LD. If $w = \lambda v$.
$\Rightarrow (v, w)$ is LD. Since $0 = \lambda \cdot v + (-1) \cdot w$. $\Rightarrow$ it is LD.
Therefore, the all cases are Contradiction.

EXAMPLE: INFINITE L.I. SET.

$\{1, x, x^2, x^3, \cdots x^n, x^{n+1} \cdots\}$ in $F[x]$ F is a field. is LI.

We have check that any finite subset is LI.

$0 = a_1 x^{i_1} + a_2 x^{i_2} + a_3 x^{i_3} + \cdots + a_s x^{i_s} \iff a_1 = a_2 = \cdots = a_s.$ $i_j \neq i_k$ $j \neq k$.

since the 0 polynomial has all finite coefficients.

EXAMPLE:

$\{(1,0,1), (2,0,2)\}$ is LD. $(0,0,0) = -2(1,0,1) + (2,0,2)$.

$\Rightarrow \{(1,0,1), (2,0,2), (1,5,10), \cdots\}$ is LD.

PROBLEM: What is the connetion between L.D sets and spanning set?

Proposition: let V be an F-vector space, $\{v_1, v_2, \cdots v_n\} \subseteq V$. Then:

The list $(v_1, v_2, \ldots v_n)$ is L.D. $\iff \exists j,\ v_j \in \operatorname{span}(v_1, v_2 \ldots v_{j-1})$. for $1 \leq j \leq n$.

In this case: $\operatorname{span}(v_1, v_2, \cdots v_n) = \operatorname{span}(v_1, v_2 \cdots v_{j-1}, v_{j+1}, \cdots v_n)$.

Proof:

$\Rightarrow$) Assume $(v_1, v_2, \cdots v_n)$ is L.D. $\Rightarrow 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ $\lambda_i \in F$. st $\{k, \lambda_k \neq 0\}$ is not empty and finite. let $j = \max\{k : \lambda_k \neq 0\}$, Then:

$$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1} + \lambda_j v_j. \Rightarrow v_j = \left(\frac{-\lambda_1}{\lambda_j}\right) v_1 + \left(\frac{-\lambda_2}{\lambda_j}\right) v_2 + \left(\frac{-\lambda_3}{\lambda_j}\right) v_3 + \cdots + \left(\frac{-\lambda_{j-1}}{\lambda_j}\right) v_{j-1}.$$

$v_i \in \operatorname{Span}(v_1, v_2 \ldots v_{j-1})$.

$\Leftarrow$). if $v_j \in \operatorname{span}(v_1, v_2, \cdots v_{j-1}) \Rightarrow v_j = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1}$, $\lambda_1 \cdots \lambda_{j-1} \in F$.

$$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1} + (-1) \cdot v_j \Rightarrow \{v_1, v_2, \cdots v_j, v_{j+1} \cdots v_n\} \text{ is L.D.}$$ Q

In this case, $v_j = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1}$.

$a_1 v_1 + a_2 v_2 + \cdots + a_j v_j + \cdots + a_n v_n = a_1 v_1 + a_2 v_2 + \cdots + a_{j-1} v_{j-1} + a_j(\lambda_1 v_1 + \cdots + \lambda_{j-1} v_{j-1}) + a_{j+1} v_{j+1} + \cdots + a_n v_n$

$= (a_1 + a_j \lambda_1) v_1 + (a_2 + a_j \lambda_2) v_2 + \cdots + (a_{j-1} + a_j \lambda_{j-1}) v_{j-1} + a_{j+1} v_{j+1} + \cdots + a_n v_n$

$\Rightarrow \operatorname{span}(v_1, v_2 \cdots v_j \cdots v_n) = \operatorname{Span}(v_1, v_2 \cdots v_{j-1}, v_j, \cdots v_n)$.

Example:

$S = \operatorname{span}((1,0,0), (2,0,0), (0,1,0), (0,3,0), (0,0,1)) \subseteq \mathbb{R}^3$.

$= \{a(1,0,0) + b(2,0,0) + c(0,1,0) + d(0,3,0) + e(0,0,1),\ a, b, c, d, e \in \mathbb{R}\}$.

$j=1$ $(1,0,0) \notin \operatorname{span}(\phi) =$ SMALLEST SUBSPACE CONTAINING $\phi = \{(0,0,0)\}$. $\Rightarrow j \neq 1$.

$j=2$. $(2,0,0) \in \operatorname{span}(1,0,0)$, since $(2,0,0) = 2 \cdot (1,0,0)$. ✓.

$\Rightarrow S = \operatorname{span}((1,0,0), (0,1,0), (0,3,0), (0,0,1))$.

To the same way, consider $j = 3, 4$. $\Rightarrow$ $S = \operatorname{span}((1,0,0), (0,1,0), (0,0,1))$.

Remark:

$j=1 \iff v_1 \in \operatorname{Span}(\phi) = \{0\} \iff v_1 = 0$.

$j=2 \iff v_2 \in \operatorname{span}(v_1) \iff \{v_1, v_2\}$ L.D.

$j=3 \iff v_3 \in \operatorname{span}(v_1, v_2) \iff \{v_1, v_2, v_3\}$ L.D.

$v_3 = \lambda_1 v_1 + \lambda_2 v_2$.

Proposition: Let V be an F-vector space, $\{v_1, v_2, v_3 \cdots v_m\}$ is a LI set. $\{w_1, w_2, \cdots w_n\}$ spanning set, that is $V = \text{span}(w_1, w_2, \dots w_n)$. Then $m \le n$.

THE CARDINALITY OF ANY LI SET $\le$ CARDINALITY OF ANY SPANNING SET.

Example:

$R^3 = \{(x,y), x,y \in R\} = \{(x,y) = x(1,0) + y(0,1)\} = \text{span}((1,0),(0,1))$. $n=2$.

If $\{(a,b), (c,d), (e,f)\}$ L.D $m = 3 \nleq 2$. $\Rightarrow$ It is not a subspace of R^2.

Now, we need to prove the proposition.

let $V = \text{span}(w_1, w_2, w_3 \dots w_n)$, $B_0 = \{w_1, w_2, \dots w_n\}$, $\# B_0 = n$.

STEP 1: $v_1 \in V = \text{span}(w_1, w_2, \cdots w_n) \Leftrightarrow \{w_1, w_2 \cdots w_n, v_1\}$ L.D. $\Leftrightarrow \{v_1, w_1, w_2 \cdots w_n\}$ L.D.

$\Leftrightarrow \exists j_1, j_1 \geq 1$. SMALLEST $w_{j_1} \in \text{span}(v_1, w_1, \cdots w_{j_1-1})$. st $v_1 = \lambda \cdot w_{j_1}$ $\lambda \in F$.

$V = \text{span}(w_1, w_2, \cdots w_n) = \text{span}(v_1, w_1, \cdots w_n) = \text{span}(v_1, w_1 \cdots w_{j_1-1}, w_{j_1+1}, \cdots w_n)$

$B_1 = \{v_1, w_1 \cdots w_{j_1-1}, w_{j_1+1}, \cdots w_n\}$, $\# B_1 = n$.

STEP 2: $v_2 \in V = \text{span}(v_1, w_1, w_2 \cdots w_{j_2} \cdots w_n) \Leftrightarrow \{v_1, w_1 \cdots w_{j_2} \cdots w_n, v_2\}$ LD. $\Leftrightarrow \{v_1, v_2, w_1 \cdots w_{j_2} \cdots w_n\}$ LD.

$\Leftrightarrow \exists j_2, j_2 \geq 2$. st $w_{j_2} \in \text{span}(v_1, v_2, w_1 \cdots w_{j_2-1})$.

$\Rightarrow V = \text{span}(B_1) = \text{span}(v_1, v_2, w_1 \cdots w_{j_2} \cdots w_n) = \text{span}(v_1, v_2 \overbrace{\cdots w_{j_2-1}, w_{j_2+1} \cdots w_n}^{B_2})$.

$\vdots$

Step m: $v_m \in V = \text{span}(B_{m-1}) = \text{span}(v_1, v_2 \cdots \overbrace{v_m, w_1 \cdots w_{j_{m-1}}, w_{j_m+1}, \cdots}^{n} w_n)$.

$\Leftrightarrow \{\underset{\underset{\text{LI}}{\Downarrow}}{v_1, v_2 \cdots v_m}, \underset{\underset{\text{cannot be empty}}{\Downarrow}}{w_1, w_2 \cdots w_{j_1}, w_{j_2}, \cdots w_{j_m}, \dots w_n}\}$ LD. $\Rightarrow m < \#(\{v_m\} \cup B_{m-1}) = 1+n$

$\Leftrightarrow m \le n$.

4.2. ClASS 10.

last class: we get CARDINALITY OF ANY SPANNING SET.

$\{v_1, v_2 \dots v_m\}$ LI, $V = \text{span}(w_1, w_2, \cdots w_n) \Longrightarrow m \le n$.

ADDICATION:

All subspaces of R^2 are $\{(0,0)\}$, R^2 and $L = \{\lambda(a,b), (a,b) \in R^2, (a,b) \neq 0, \lambda \in R\}$.

First we check L are subspace: ....

let S be a subspace.

S subspace $\Rightarrow (0,0) \in S \Rightarrow \{(0,0)\} \subseteq S$. If $\{(0,0)\} = S \Rightarrow$ We have done.

If $\{(0,0)\} \subsetneq S \Rightarrow \exists (a,b) \in R^2, (a,b) \neq (0,0) : (a,b) \in S$.

S subspace, $(a,b) \in S \Rightarrow \lambda(a,b) \in S, \forall \lambda \in R \Rightarrow L = \{\lambda(a,b), \lambda \in R\} \subseteq S$.

If $S = \{\lambda(a,b), \lambda \in R\}$ We have done.

If $\{\lambda(a,b), (a,b) \neq (0,0), \lambda \in R\} \subsetneq S \Rightarrow \exists (c,d) \in S$. $(c,d) \neq \lambda(a,b)$ $\forall \lambda \in R$.

$\Rightarrow \{(a,b), (c,d)\}$ is L.I.

$R^2 = \{(x,y) = x(1,0) + y(0,1), x,y \in R\} = \text{span}((1,0),(0,1))$. $\text{span}((1,0),(0,1))$. LI.

$\Rightarrow (x,y) \in \text{span}((a,b),(c,d)) = \{\lambda_1(a,b) + \lambda_2(c,d), \lambda_1, \lambda_2 \in R\} \subseteq S$.

$\Rightarrow R^2 \subseteq S \Rightarrow R^2 \subseteq S \subseteq R^2 \Rightarrow S = R^2$.

THEO: LET V BE A FINITE-DIMENTIONAL VECTOR SPACE, AND LET S BE A SUBSPACE OF V. THEN S IS ALSO A FINITE DIMENTIONAL VECTOR SPACE.

Proof: V finite-dimentional $\Leftrightarrow \exists$ finit set $X \subseteq V : V = \text{span}(X)$, $X = \{w_1, w_2, \dots w_n\}$

If $S = \{0\} = \text{span}\{\emptyset\} \Rightarrow S$ is finite-dimentional.

if not $\exists v_1 \in S$ $v_1 \neq 0$.

if $S = \text{span}(v_1) \Rightarrow S$ is finite-dimentional

If not, $\text{span}(v_1) \subsetneq S \Rightarrow \exists v_2 \in S$, $v_2 \notin \text{span}(v_1) \Longrightarrow \{v_1, v_2\}$ L.I.

If $S = \text{span}(v_1, v_2) \Rightarrow S$ is FD.

If not $\text{span}(v_1, v_2) \subsetneq S$, $\exists v_3 \in S$, $v_3 \notin \text{span}(v_1, v_2)$. $\Rightarrow \{v_1, v_2, v_3\}$ L.I.

Since the CARDINALITY of any L.I. set is $\leq n$. This algorial should stop in step k:

$S = \text{span}(v_1, v_2 \dots v_k) \Rightarrow S$ is F-D.

BASIS.

Def:

A basis for a vector space V is an ordered L.I. spanning set. That is:

B is a basis $\Leftrightarrow B$ is L.I. and $V = \text{span}(B)$.

REMARK: $B = \{v_1, v_2, \dots v_n\}$ $V = \text{span}(v_1, v_2 \dots v_n)$. $\Leftrightarrow$ ANY $v \in V$ CAN BE WRITTEN AS A L.C. OF $v_1, v_2 \dots v_n$: $v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = \mu_1 v_1 + \mu_2 v_2 + \dots + \mu_n v_n \Leftrightarrow 0 = (\lambda_1 - \mu_1) v_1 + (\lambda_2 - \mu_2) v_2 + \dots + (\lambda_n - \mu_n) v_n$.

B is L.I. $\Leftrightarrow v$ can be written in a UNIQUE WAY.

Example:

1) $R^3 = \{(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1) , x, y, z \in R\}$.
$\Downarrow$
can be written in a UNIQUE WAY.

$\Rightarrow E_3 = e_3 = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis.

$B = \{(0,1,0), (0,0,1), (1,0,0)\}$ is another basis.

$(x,y,z) = y(0,1,0) + z(0,0,1) + x(1,0,0)$

2) $S = \{(x, x+y, x), x, y \in R\}$ is a subspace.

Find a basis:

$(x, x+y, x) = (x, x, x) + (0, y, 0) = x(1,1,1) + y(0,1,0)$. $S = \text{span}((1,1,1), (0,1,0))$. $\rightarrow$ spanning list.

$a(1,1,1) + b(0,1,0) = (0,0,0) \Rightarrow a = b = 0$. $\Rightarrow$ L.I. SET.

$B = \{(1,1,1), (0,1,0)\}$ is a basis for S.

3) $R_3[x] = \{p(x) \in R[x] : p(x) = 0 \text{ or } \deg(p(x)) \leq 3\} = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3, a_0, a_1, a_2, a_3 \in R\}$.

$= \text{span}(1, x, x^2, x^3) \longrightarrow$ spanning list.

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0 \Leftrightarrow a_0 = a_1 = a_2 = a_3 = 0$ L.I.

$B = \{1, x, x^2, x^3\}$ is a basis of $R_3[x]$.

4) $S = \text{span}\left((1,0,0), (2,0,0), (1,1,0), (2,2,0)\right) = \text{span}\left((1,0,0), (1,1,0), (2,2,0)\right) = \text{span}\left((1,0,0) (1,1,0)\right)$.

So $B = \{(1,0,0), (1,1,0)\}$ is a basis of S.

Problem: HOW CAN WE CONSTRUCT BASIS FOR VECTOR SPACES ?

THEO 1: any spanning list of V contains a basis.

$V = \mathrm{span}(X) \Rightarrow \exists B$ basis, $B \subseteq X$.

THEO 2: any L.I. set can be extended to a basis.

$Y \subseteq V$. Y is LI. $\Rightarrow \exists B$ basis, $Y \subseteq B$.

Proof: (we will write the proof only for finite-dimentional vector spaces).
(Not Finite-DIMENTIONAL, we use ZORN's LEMMA).

(1). let V is F-D. X is finite, $X = \{w_1 w_2 \dots w_n\} = B_0$.

STEP 1. If $w_1 = 0$, $B_1 = B_0 \setminus \{w_1\}$. If $w_1 \neq 0$ $B_1 = B_0$. $\Rightarrow \mathrm{span}(B_1) = \mathrm{span}(B_0)$.

STEP 2. if $w_2 \in \mathrm{span}(B_1) \Rightarrow B_2 = B_1 \setminus \{w_2\}$ if $w_2 \notin \mathrm{span}(B_1) \Rightarrow B_2 = B_1 \Rightarrow \mathrm{span}(B_2) = \mathrm{span}(B$

• • •

STEP n : If $w_n \in \mathrm{span}(B_{n-1}) \Rightarrow B_n = B_{n-1} \setminus \{w_n\}$. $\Rightarrow B_n$ is a basis of V.
if $w_n \notin \mathrm{span}(B_{n-1}) \Rightarrow B_n = B_{n-1}$.

$V = \mathrm{span}(X) = \mathrm{span}(B_0) = \mathrm{span}(B_1) = \cdots = \mathrm{span}(B_n)$.

We can use INDUCTION to prove it.

$\mathrm{span}(B_n) = \mathrm{span}(B_0) = \mathrm{span}(X)$. $\forall n$.

if $n=1$. $\mathrm{span}(B_0) = \mathrm{span}(B_1)$. IH. $\Rightarrow B_n$ is a spanning list.

Take $n-1$. $\Rightarrow \mathrm{span}(B_0) = \mathrm{span}(B_{n-1}) \overset{\checkmark}{=} \mathrm{span}(B_n)$.

Now, need to prove B_n is L.I.

Assume B_n is LD. $B_n = \{v_1 v_2 \dots v_s\}$ LD $\Rightarrow \exists v_j$ st $v_j \in \mathrm{span}(v_1, v_2 \cdots v_{j-1}\ v_{j+1} \dots v_n)$.

But $v_j = w_k$. $w_k \in \mathrm{span}(v_1, \dots v_{j-1}) \subseteq \mathrm{span}(B_{k-1})$ CONTRADICTION to step k.

$\Rightarrow B_n$ LI.

COROLLARY: any vector space admit a basis.

Proof: $V = \mathrm{span}(V) \Rightarrow \exists B \subseteq V$, B is basis.

Now we prove theo 2.

V is F-D. $\Rightarrow \exists X$ finite : $V = \mathrm{span}(X)$ If Y is L.I. $\#Y \leq \#X$.

$\Rightarrow Y$ is finite.

If $V = \mathrm{span}(Y) \Rightarrow Y$ is spanning set and LI. $\Rightarrow B = Y$ Basis.

if not, $\mathrm{span}(Y) \subsetneq V$, $\exists v_1 \in V$ $v_1 \notin \mathrm{span}(Y)$. $\Rightarrow B_1 = Y \cup \{v_1\}$ Y is LI.

If $V = \mathrm{span}(B_1) \Rightarrow Y \subseteq B_1$ B_1 is basis.

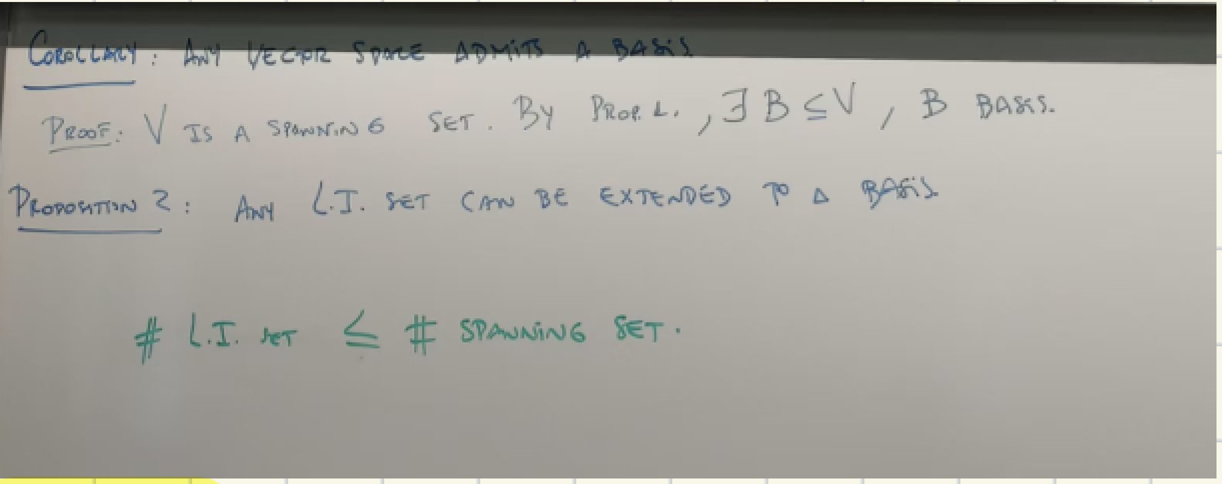
If not, $\mathrm{span}(B_1) \subsetneq V \Rightarrow \exists v_2 \in V$. $v_2 \notin \mathrm{span}(B_1) \Rightarrow B_2 = Y \cup \{v_1, v_2\}$ Y is L.I.

This has to finish in a finite number of steps since CARDINALITY of LI. sets $\leq \#X$.

↓ Stop at step k: $V = \mathrm{span}(B_k)$, B_k L.I. and spanning list $\Rightarrow Y \subseteq B_k$. B_k basis.

4.7. ClASS 11.

Proposition: Any spanning set contains a basis.



COROLLARY: ANY VECTOR SPACE ADMITS A BASIS

PROOF: V IS A SPANNING SET. BY PROP. 1., $\exists B \subseteq V$, B BASIS.

PROPOSITION 2: ANY L.I. SET CAN BE EXTENDED TO A BASIS

$\#$ L.I. SET $\leq$ $\#$ SPANNING SET.

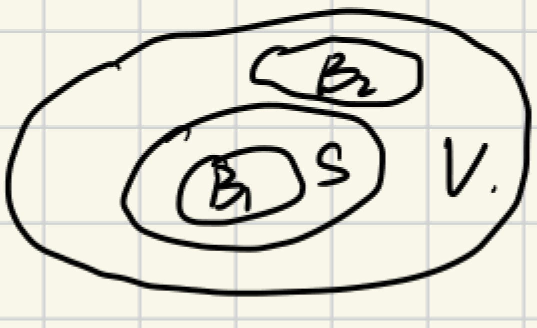
Corollary: Any subspace S of a vector space V admits a **COMPLEMENT**. That is, $\exists T \leq V$, a subspace st $V = S \oplus T$. (any $v \in V$ can be written as the addition $v = v_1 + v_2$, $v_1 \in S$, $v_2 \in T$.)

THEO:

S is a subspace of $V \Rightarrow S$ is a vector space $\Rightarrow \exists B_1$ a basis of S. $\Rightarrow B_1$ is LI in S.

$\Rightarrow B_1$ is LI in V. $\Rightarrow B_1 \subseteq S$. let B a basis of V.

$B = B_1 \cup B_2$. $B_2 = B \setminus B_1$.



let $T = \text{span}(B_2)$ let's prove that $V = S \oplus T$.

$S + T = \text{span}(B_1) + \text{span}(B_2) = \text{span}(B_1 \cup B_2) = \text{span}(B) = V$.

$S \cap T = \{0\}$: Since $\{0\} \subseteq S \cap T$.

$v \in S \cap T \Rightarrow v \in S \wedge v \in T$, set $B_1 = \{v_i, i \in I\}$. $B_2 = \{w_j, j \in J\}$. $B = B_1 \cup B_2$. is a basis $\Rightarrow B_1 \cap B_2 = \phi$.

$v = a_1 v_1 + \cdots + a_k v_k = b_1 w_1 + \cdots + b_m w_m$. $\Rightarrow 0 = \underbrace{a_1 v_1 + \cdots + a_k v_k - b_1 w_1 - \cdots - b_m w_m}_{= B_1 \cup B_2.}$. Since $B_1 \cup B_2$ is LI.

$\Rightarrow a_1 = a_2 = \cdots = b_1 = b_2 = \cdots = b_m$. $\Rightarrow v = 0$.

DIMENTION

THEO: let B_1, B_2 be two basis of the F-vector space V. Then $\#B_1 = \#B_2$.

Proof: Basis = LI + spanning set.

B_1 LI, B_2 spanning set $\Rightarrow \#B_1 \leq \#B_2$.

B_2 LI, B_1 spanning set $\Rightarrow \#B_1 \geq \#B_2$. $\Rightarrow$ Then $\#B_1 = \#B_2$.

Def: $\dim_F V = \#B$. For B any basis for V, called the dimention of V.

Example.

1) F^n, $\{e_1, e_2, \dots e_n\}$ is a basis $e_i = (0, 0, \dots 1, 0, 0 \dots 0)$. $\Rightarrow \dim_F F^n = n$.

2) $M_{n \times m}(F)$, $\{E_{ij}, 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis $E_{ij} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$. $\dim_F M_{n \times m}(F) = n \times m$.

3) $\dim_{\mathbb{C}} \mathbb{C}^2 = ?$ $\dim_{\mathbb{R}} \mathbb{C}^2 = ?$

$\mathbb{C}^2 = \text{span}((1,0), (0,1))$, L.I.

$\Rightarrow \dim_{\mathbb{C}} \mathbb{C}^2 = 2$.

$\mathbb{C}^2 = \text{span}((1,0), (i,0), (0,1), (0,i))$ $(a+bi, c+di) = a(1,0) + b(i,0) + c(0,1) + d(0,i)$.

$\{(1,0), (0,1), (i,0), (0,i)\}$ is LI in $\mathbb{R}$.

$\Rightarrow \dim_{\mathbb{R}} \mathbb{C}^2 = 4$.

THEO 1: If S is a subspace of V. Then $\dim_F S \leq \dim_F V$.

Proof: let B_1 be a basis of S. B_2 a basis of V.

B_1 is LI in $S \Rightarrow B_1$ is LI in V.

B_2 is a spanning set for V. $\Rightarrow \#B_1 \leq \#B_2 . \Rightarrow \dim_F S \leq \dim_F V$.

THEO: let V be an F-vector space of finite dimention, that is $\dim_F V = n$. let $B \subseteq V$. $\dim_F V = n$. then following are equivalent: $\#B = n$

a) B is a basis of V.

b) B is LI in V.

c) B is a spanning set for V.

Remark: false for infinite dimention:

$\{x, x^2, \dots\}$ is LI in $R[x]$ but is not a spanning set.

Proof:

a) $\Rightarrow$ b) ✓. Basis $\Leftrightarrow$ LI + spanning $\Rightarrow$ LI.

b) $\Rightarrow$ c) = B is LI in V, $\#B = n$. $\exists B_1$ basis: $B \subseteq B_1$ $\#B = n = \dim_F V = \#B_1$

$\Rightarrow B = B_1 \Rightarrow B$ spanning set.

c) $\Rightarrow$ a): B is a spanning set $\Rightarrow \exists B_2$ basis : $B_2 \subseteq B$.

$\#B_2 = \dim_F V = n = \#B. \Rightarrow B_2 = B \Rightarrow B$ is a basis.

Example:

i) $\dim_{\mathbb{C}} \mathbb{C}^2 = 2$. is $\{(1+i, 2), (2i, 3)\}$ a basis?

$(a+bi)(1+i, 2) + (c+di)(2i, 3) = ((a-b)+(a+b)i, 2a+2bi) + (-2d+2ci, 3c+3di)$

$= ((a-b-2d)+(a+b+2c)i, 2a+3c+(2b+3d)i) = (0,0) \Leftrightarrow \begin{cases} a-b-2d=0 \\ a+b+2c=0 \\ 2a+2c=0 \\ 2b+3d=0 \end{cases} \Rightarrow a=b=c=d=0 \Rightarrow$ is basis.

2) $B = \{(1, a_2, a_3), (0, 1, b_3), (0,0,1)\}$ is a basis F^3 since $\dim_F F^3 = 3$.

$\lambda_1 (1, a_2, a_3) + \lambda_2 (0, 1, b_3) + \lambda_3 (0,0,1) = (0,0,0)$

$\Rightarrow (\lambda_1, \lambda_1 a + \lambda_2, \lambda_1 a_3 + \lambda_2 b_3 + \lambda_3) = (0,0,0) \Leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$.

B is LI, $\#B = 3 \Rightarrow B$ is a basis.

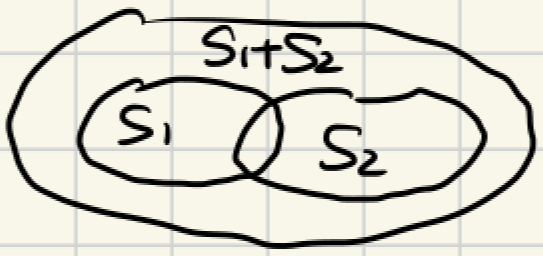
3) $B = \{1+ax+bx^2, x+cx^2, x^2\}$ LI in $R[x]_{\leq 2}$. $\dim_R R[x]_{\leq 2} = 3$.

$\Rightarrow B$ is a basis.

PROBLEM: Compute $\dim V = ?$

THEO: If $V = S_1 + S_2$, $\dim V < \infty$ Then $\dim(S_1+S_2) = \dim S_1 + \dim S_2 - \dim(S_1 \cap S_2)$

In particular: $\dim(S_1 \oplus S_2) = \dim S_1 + \dim S_2$.



Proof:

let $\{v_1, v_2 \cdots v_r\}$ be a basis for $S_1 \cap S_2 \Rightarrow \{v_1, v_2 \cdots v_r\}$ is LI in S_1 and S_2.

We can extend this set to basis for S_1 and S_2.

$B_1 = \{v_1 \dots v_r, u_1, \dots u_s\}$ basis for S_1. $\rightarrow \dim S_1 = r+s$. $\Rightarrow$ We have to prove that:

$B_2 = \{v_1 \cdots v_r, w_1 \cdots w_k\}$ basis for S_2. $\rightarrow \dim S_2 = r+k$. $\dim S_1 + S_2 = r+s+r+k-r = r+s+k$.

let's prove that $B = \{v_1 \cdots v_r, w_1 \cdots w_k, u_1 \cdots u_s\}$ is a basis of S_1+S_2.

$\mathrm{span}(B) = \mathrm{span}(B_1 \cup B_2). = \mathrm{span}(B_1) + \mathrm{span}(B_2). = S_1 + S_2. \Rightarrow B$ is a spanning set.

B is LI:

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in S_1 \qquad\qquad\qquad\qquad \in S_2.$

$0 = a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s + c_1w_1 + \cdots + c_kw_k. \Leftrightarrow a_1v_1 + \cdots + a_rv_r + b_1u_1 + \cdots + b_su_s = (-c_1)w_1 + \cdots + (-c_k)w_k$

$= v \in S_1 \cap S_2. \Rightarrow (-c_1)w_1 + (-c_2)w_2 + \cdots (-c_k)w_k \in S_1 \cap S_2. = \lambda_1v_1 + \lambda_2v_2 + \cdots \lambda_rv_r$ Since

$\{v_1, \cdots v_r\}$ is a basis of $S_1 \cap S_2$. So $0 = \lambda_1v_1 + \lambda_2v_2 + \cdots + \lambda_rv_r + c_1w_1 + c_2w_2 + \cdots + c_kw_k$. B_2 is LI.

$\Rightarrow c_1 = c_2 = \cdots = c_k = \lambda_1 = \cdots = \lambda_r = 0. \Rightarrow a_1v_1 + a_2v_2 + \cdots + a_rv_r + b_1u_1 + b_2u_2 + \cdots + b_su_s = 0.$ B_1 is LI.

$\Rightarrow a_1 = \cdots = a_r = b_1 = \cdots = b_s = 0.$

So B is LI.

4.9. CLASS 12.

Question: $S_1 + S_2 = ?$

$S_1 + S_2 \subseteq R^4$. $\dim S_1 + S_4 = 4 \Rightarrow R_1 + R_2 = R^4$.

THEO: If $S \subseteq V$. subspace, $\dim V < \infty \wedge \dim S = \dim V \Rightarrow S = V$.

Proof: let B be a basis of S. $\Rightarrow$ B is LI in S. $\Rightarrow$ B LI in V.

$\# B = \dim S = \dim V. \Rightarrow$ B is a basis in V. $\Rightarrow S = \mathrm{span}(B) = V$.

Remark: THE PREVIOUS THEO IS NOT TRUE WITHOUT THE ASSUMPTIONS:

1) $S \nsubseteq V$. $S = \{(x,0), x \in R\}$, $V = \{(0,y), y \in R\}$, $\dim S = \dim V = 1$. But $S \neq V$.

2) $\dim V = \infty$: $S = \{p(x) \in R(x) : p(0) = a_0 = 0\} \subsetneq R[x] = V = \{1, x, x^2, \ldots\}$. $\dim S = \dim V = \infty$.
But $S \neq V$

3) $\dim S \neq \dim V$: $S = \{(x,0) : x \in R\} \subsetneq R^2$. $\dim R^2 = 2 < \infty$. But $S \neq R^2$.

Matrices.

$M_{n\times m}(F)$ = set of matrices of order $n \times m$ with coefficients in F.

$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ A_{n1} & \cdots & a_{nm} \end{pmatrix}$ = matrix in $M_{n\times m}(F)$ = set of $n \times m$ elements in F. Arranged in m ROWS. and n COLUMNS.

A_{ij} = coefficient in ROW i, COLUMN j.

$R_i = R_i(A) = i$-ROW $= [A_{i1}\ A_{i2}\ A_{i3} \cdots A_{im}]$. $C_j = C_j(A) = \begin{bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{nj} \end{bmatrix}$

$A \in M_{n\times m}(F)$ then $A = B \Leftrightarrow (n,m) = (s,t)$. and $A_{ij} = B_{ij}$ $\forall i \in [1,n]$ $\forall j \in [1,m]$.

Operations:

ADDITIONS: $M_{n\times m}(F) \times M_{n\times m}(F) \xrightarrow{+} M_{n\times m}(F)$.

$(A, B) \longrightarrow A + B = (A+B)_{ij} = A_{ij} + B_{ij}$.

PROPERIES: (S_1) ASSOCIATIVE. (S_2) COMMUTATIVE.

(S_3) $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

(S_4) $-\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} -a_{11} & \cdots & -a_{1m} \\ \vdots & & \\ -a_{n1} & \cdots & -a_{nm} \end{pmatrix}$.

Product by scalars: $F \times M_{n\times m}(F) \longrightarrow M_{n\times m}(F)$.

$(\lambda, A) \longrightarrow (\lambda A). = \lambda \cdot A_{ij}$.

Properies:

1) $\lambda\cdot(A+B) = \lambda A + \lambda B \in M_{n\times m}(F)$.

$(\lambda(A+B))_{ij} = \lambda\cdot(A+B)_{ij} = \lambda(A_{ij}+B_{ij}) = \lambda\cdot A_{ij} + \lambda\cdot B_{ij} = (\lambda A+\lambda B)_{ij}$. $\forall ij$.

2) $(\lambda+\mu)A = \lambda A + \mu A$.

3) $(\lambda\cdot\mu)A = \lambda(\mu A)$.

4) $1_F \cdot A = A$. $\forall A$.

$\Rightarrow$ $M_{n\times m}(F)$ is F-vector space:

$M_{n\times m}(F) \times M_{m\times s}(F) \longrightarrow M_{n\times s}(F)$. (if $n=m=s$: $M_n(F) = M_{n\times n}(F)$.).

$M_n(F)\times M_n(F) \xrightarrow{\cdot} M_n(F)$

$(A,B) \xrightarrow{\cdot} A.B$. $(A\cdot B)_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$. $= \langle R_i(A), C_j(B)\rangle$.

Properties:

P1. **Associativity:** $(\overset{n\times m}{A}\cdot\overset{m\times s}{B})\cdot\overset{s\times t}{C} = \overset{n\times m}{A}\cdot(\overset{m\times s}{B}\cdot\overset{s\times t}{C})$ $((A\cdot B)\cdot C)_{ij} = \cdots = (A\cdot(B\cdot C))_{ij}$.

P3. **Identity:** $\overset{n\times n}{I_n}\cdot\overset{n\times m}{A} = \overset{n\times m}{A} = \overset{n\times m}{A}\cdot\overset{m\times m}{I_m}$.

$I_s = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & 1 & \vdots \\ \vdots & \ddots & \vdots \\ 0, 0 & \cdots & 1 \end{bmatrix}$ (I_n $M_n(F)$, $I = I_n$).

Distributivity: $\overset{n\times m}{A}\cdot(\overset{m\times s}{B}+\overset{m\times s}{C}) = \overset{n\times m}{A}\cdot\overset{m\times s}{B} + \overset{n\times m}{A}\cdot\overset{m\times s}{C}$.

1) $\lambda\cdot(A\cdot B) = (\lambda\cdot A)\cdot B = A\cdot(\lambda B)$.

2) $\lambda\cdot A = (\lambda\cdot Id)A = A\cdot(\lambda\cdot Id)$.

Remark: (P2) Commutivity is not true.

$a\neq b$: $\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\cdot\begin{pmatrix} x & y \\ z & t\end{pmatrix} = \begin{pmatrix} ax & ay \\ bz & bt\end{pmatrix}$. $\begin{pmatrix} x & y \\ z & t\end{pmatrix}\cdot\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} = \begin{pmatrix} ax & by \\ az & bt\end{pmatrix}$ not equal.

P4 A. $\exists A^{-1}$: $A\cdot A^{-1} = Id$. NOT TRUE.

$\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} = I_2 = \underset{A}{\begin{pmatrix} 1 & 1 \\ 0 & 0\end{pmatrix}}\underset{A^{-1}}{\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2\end{pmatrix}} = \begin{pmatrix} x_1 & x_2 \\ 0 & 0\end{pmatrix} \neq I_2$. $\Rightarrow \nexists A^{-1}$.

$A\cdot B = 0 \not\Rightarrow A=0$ or $B=0$.

$\underset{\neq 0}{\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}}\cdot\underset{\neq 0}{\begin{pmatrix} 1 & 2 \\ 0 & 0\end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}$.

ELEMENTARY OPERATIONS ON ROWS. = ELEMENTARY MATRICES.

element operations:

1) $R_i \leftrightarrow R_j$ interchange Rows.

2) $R_i \longrightarrow a \cdot R_i$ $a \neq 0$. multiplication with a.

3) $R_i \longrightarrow R_i + aR_j$ $i \neq j$. Replace R_i by $R_i + aR_j$.

1) $P_{ij} = \begin{array}{c} \begin{bmatrix} 1 & \cdots & & & 0 \\ 0 & \ddots & & & \\ \vdots & & 0 & \cdots\ 1 & \\ & & 1 & \ 0 & \\ 0 & \cdots & \vdots & \vdots & 1 \end{bmatrix} \\ \quad\ \ i \quad j \end{array}$ $R_i \leftrightarrow R_j$: $P_{ij} \times A$. $Id \longrightarrow P_{ij}$.

$C_i \leftrightarrow C_j$: $A \times P_{ij}$. $Id \longrightarrow P_{ij}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =$$

2) $M_i(a) = \begin{bmatrix} 1 & \cdots & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & a & \vdots \\ & & & \ddots \\ 0 & \cdots & & 1 \end{bmatrix}$ $R_i \longrightarrow a \cdot R_i$ $M_i(a) \cdot A$. $Id \longrightarrow M_i(a)$

$C_i \longrightarrow a \cdot C_i$ $A \cdot M_i(a)$. $Id \longrightarrow M_i(a)$. ⊗

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & a \\ 0 & 0 & 1 \end{pmatrix}.$$

3) $T_{ij}(a) = \begin{array}{r} \\ i \\ j \\ \\ \end{array}\begin{array}{c} \begin{bmatrix} 1 & \cdots & & & 0 \\ \vdots & 1 & \cdots & a & \vdots \\ & & \ddots & 1 & \\ 0 & \cdots & & & 1 \end{bmatrix} \\ \quad i \quad\ j \end{array}$ $R_i \longrightarrow R_i + aR_j$. $T_{ij}(a) \cdot A$. $Id \longrightarrow T_{ij}(a)$

$C_j \longrightarrow C_j + aC_i$. $A \cdot T_{ij}(a)$ $Id \longrightarrow T_{ij}(a)$.

$R_i = (a_{i1}, a_{i2} \cdots 0$.

$R_j = (a_{j1}, a_{j2}$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Properties:

1) $P_{ij} \cdot P_{ij} = P_{ij}(P_{ij} \cdot Id) = Id$.

$\Rightarrow P_{ij} \cdot P_{ij} \cdot A = A$.

2) $M_i(a) \cdot M_i(a^{-1}) = Id$.

3) $T_{ij}(-a) \cdot T_{ij}(a) = Id$.

Example to prove the properties:

$Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xleftrightarrow[R_2 \leftrightarrow R_3]{} P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$: $P_{23} \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ ⤸.

$Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2 \to R_2 + 5R_1]{} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: $T_{21}(5) \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{11}+a_{21} & 5a_{12}+a_{22} & 5a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$P_{ij} \cdot P_{ij} \cdot A = P_{ij} \cdot P_{ij} \cdot \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{bmatrix} = P_{ij} \cdot \begin{bmatrix} R_1 \\ \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \end{bmatrix} = \begin{bmatrix} R_1 \\ \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{bmatrix}.$$

$$T_{23}(-4) \cdot T_{23}(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$R_2 \longrightarrow R_2 + (-4)R_3$ $R_2 + (-4)R_3 + 4R_3 \longrightarrow R_2$.

$$A \cdot T_{23}(4) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13}+4a_{12} \\ a_{21} & a_{22} & a_{23}+4a_{22} \\ a_{31} & a_{32} & a_{33}+4a_{32} \end{pmatrix}.$$

Def: $A, B \in M_{n\times m}(F)$. We say that A and B are ROW-EQUIVALENT if there exists a finite number of ELEMENTARY MATRICES, $E_1 E_2 E_3 \dots E_s$. st

$$E_s \cdots E_1 \cdot A = B.$$

Remark: THIS RELATION IS AN EQUIVALENCE RELATION.

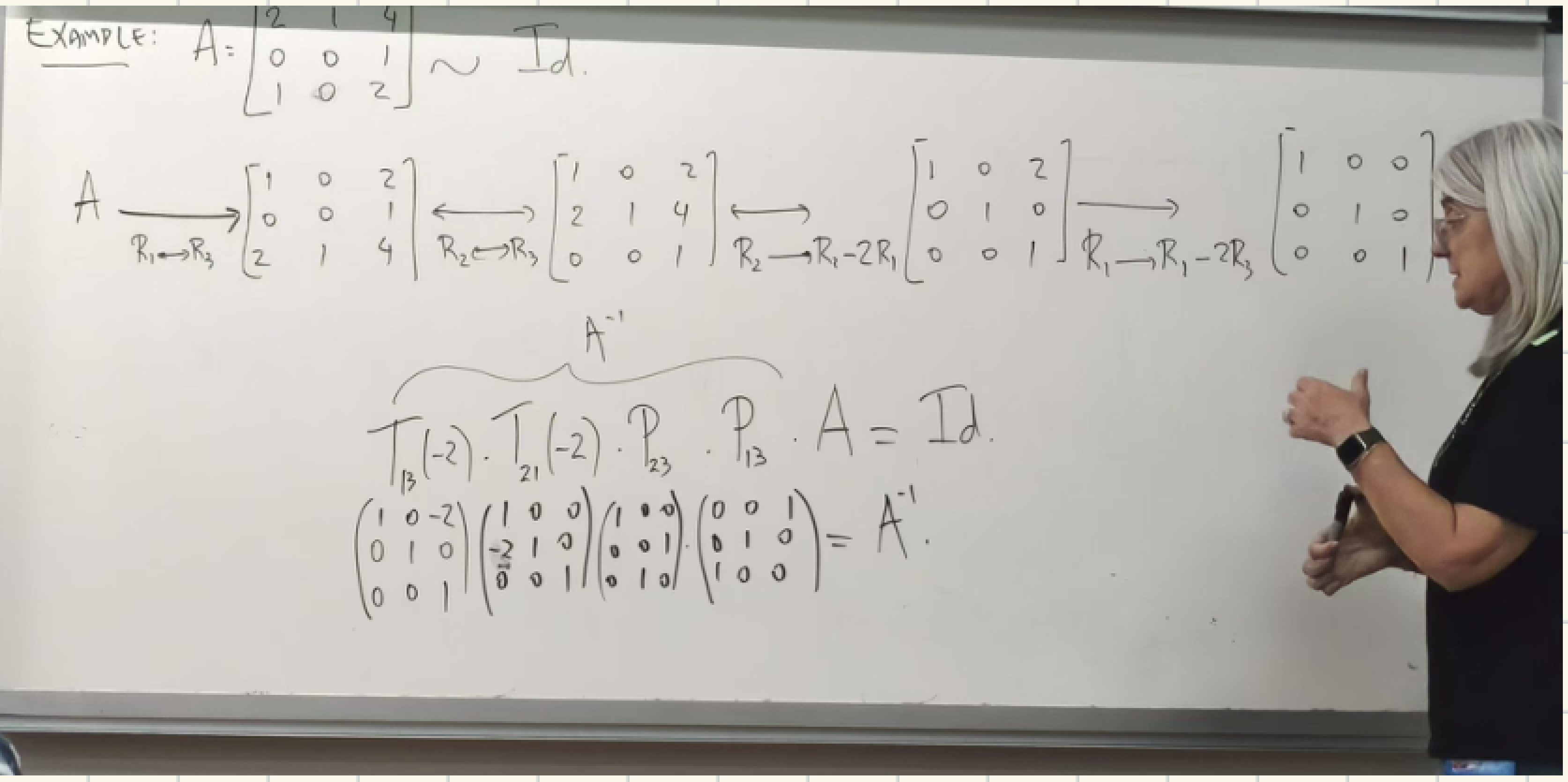
Reflexive: $Id = M_i(1)$: $A \sim A$ since $M_i(1)\cdot A = Id \cdot A = A$.

SYMMETRIC: $P_{ij}^{-1} = P_{ij}$ $M_i(a)^{-1} = M_i(a^{-1})$, $T_{ij}(a)^{-1} = T_{ij}(-a)$.

$$A \sim B \Leftrightarrow E_s \cdots E_1 A = B \Leftrightarrow A = E_s^{-1} \cdots E_1^{-1} B \Leftrightarrow B \sim A.$$

TRANSITIVITY. $A \sim B$, $B \sim C \Leftrightarrow E_s \cdots E_1 \cdot A = B.$ $E_k \cdots E_1' \cdot B = C. \Rightarrow E_k \cdots E_1' \cdot E_s \cdots E_1 \cdot A = C.$

$\Leftrightarrow A \sim C.$



EXAMPLE: $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim Id.$

$$A \xrightarrow[R_1 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 4 \end{bmatrix} \xleftrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \xleftrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 2R_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overbrace{T_{13}(-2) \cdot T_{21}(-2) \cdot P_{23} \cdot P_{13}}^{A^{-1}} \cdot A = Id.$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A^{-1}.$$

CLASS 13.

ROW\COLUMN.

1) Interchange rows R_i and R_j.
2) Multiply R_i by a non-zero scalar of $\alpha \in F$.
3). Replace R_i by $R_i + aR_j$. $i \neq j$.

(Row) ELEMENTARY MATRICES.

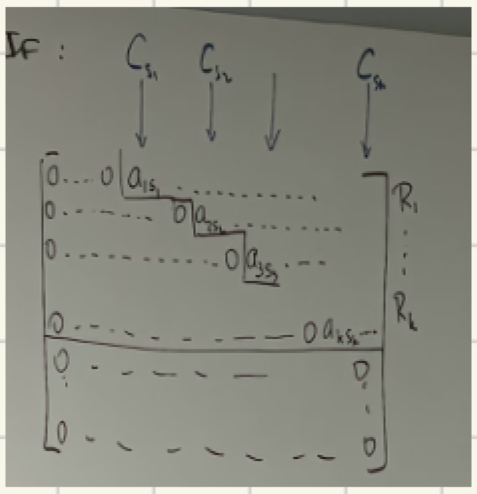
1) $Id \xrightarrow{R_i \leftrightarrow R_j} P_{ij} \xrightarrow{R_j \leftrightarrow R_i} Id.$ $A \to P_{ij} \cdot A.$
2) $Id \xrightarrow{R_i \to R_i \cdot a} M_i(a) \xrightarrow{\cdot a^{-1}} Id.$ $A \to M_i(a) \cdot A.$
3) $Id \xrightarrow{R_i \to R_i + aR_j} T_{ij}(a). \xrightarrow{-a \cdot R_i} Id.$ $A \to T_{ij}(a) \cdot A.$

COL. $A \to A \cdot P_{ij}$. $A \to A \cdot M_i(a)$. $A \to A \cdot T_{ij}(a)$.

Def: $A.B \in M_{n\times m}(F)$ are row equivalent if $\exists E_1, E_2 \dots E_s$ elementary matrices. st $E_1 E_2 \cdots E_s \cdot A = B.$

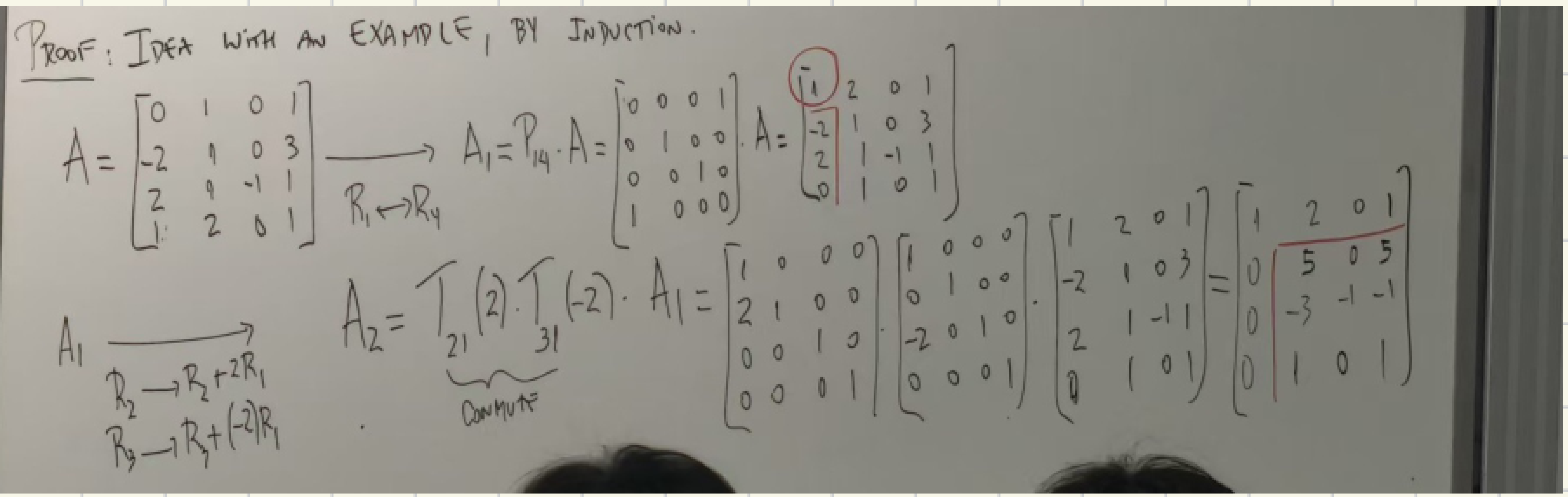
Def: A matrices $A \in M_{n\times m}(F)$ is called Row Echelon if:

1) The non-zero appears first. $R_1, R_2 \cdots R_k$ non-zero. $R_{k+1} \cdots R_n$ is 0.
2) If the first non-zero element in R_i appears in Column S_i. Then $S_1 < S_2 \cdots < S_k$
3) $a_{is_i} = 1$. $1 \leq i \leq k$.



Def: A matrix A is called Row-Reduced Echelon if it is Row Echelon and each column C_{s_i} has all its elements equal 0 except $a_{is_i}=1$.

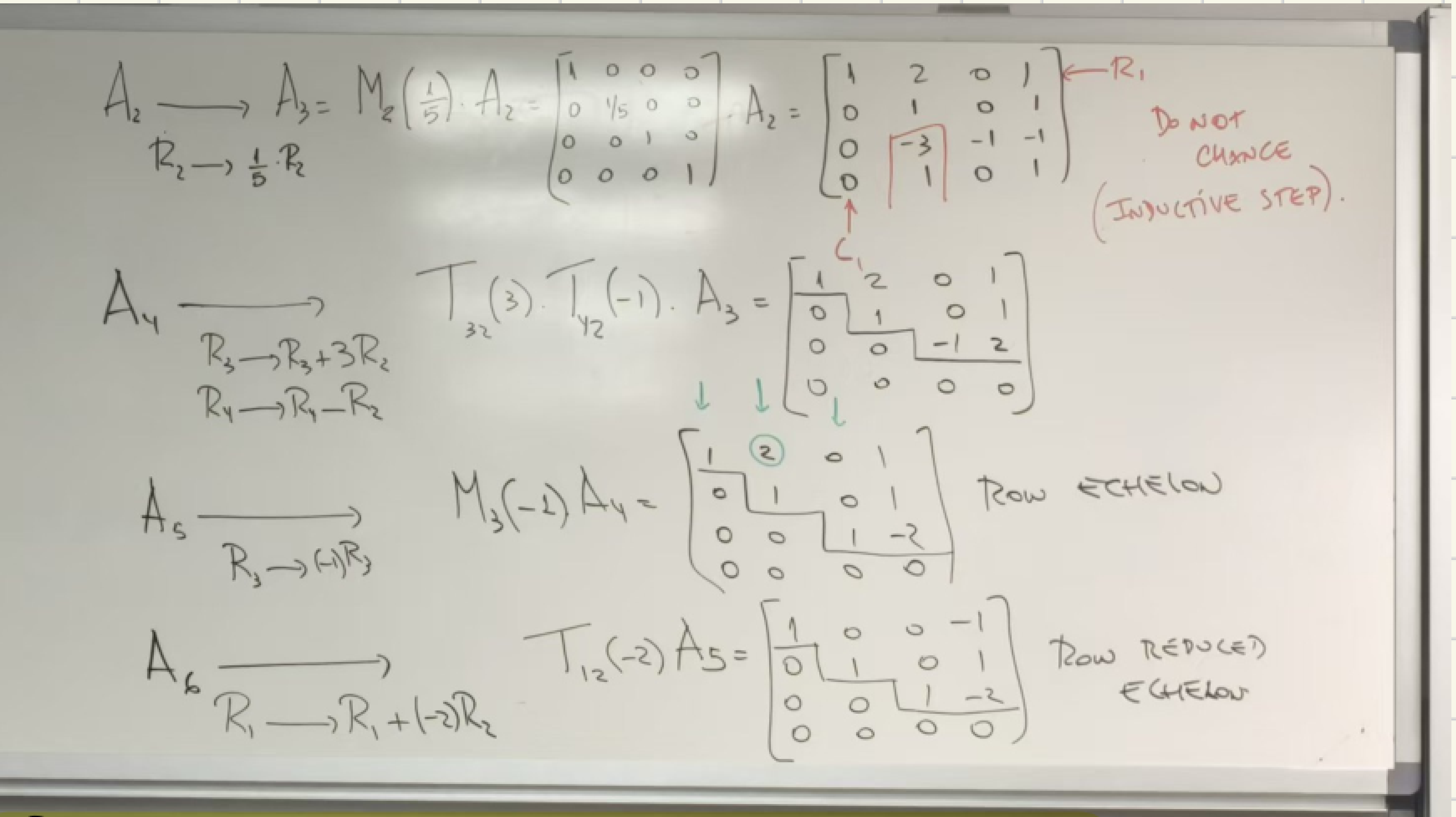
THEO: If $A \in M_{m\times m}(F)$, then exists $E_1 \cdot E_2 \cdots E_k$ elementary matrices st. $E_1 \cdots E_k A$ is Row-Reduced Echelon.



PROOF: IDEA WITH AN EXAMPLE, BY INDUCTION.

$$A=\begin{bmatrix}0&1&0&1\\-2&1&0&3\\2&1&-1&1\\1&2&0&1\end{bmatrix}\xrightarrow[R_1\leftrightarrow R_4]{} A_1=P_{14}\cdot A=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}\cdot A=\begin{bmatrix}\boxed{1}&2&0&1\\-2&1&0&3\\2&1&-1&1\\0&1&0&1\end{bmatrix}$$

$$A_1\xrightarrow[\substack{R_2\to R_2+2R_1\\ R_3\to R_3+(-2)R_1}]{} A_2=\underbrace{T_{21}(2)\cdot T_{31}(-2)}_{\text{COMMUTE}}\cdot A_1=\begin{bmatrix}1&0&0&0\\2&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&0&0&0\\0&1&0&0\\-2&0&1&0\\0&0&0&1\end{bmatrix}\cdot\begin{bmatrix}1&2&0&1\\-2&1&0&3\\2&1&-1&1\\0&1&0&1\end{bmatrix}=\begin{bmatrix}1&2&0&1\\0&5&0&5\\0&-3&-1&-1\\0&1&0&1\end{bmatrix}$$



$$A_2\xrightarrow[R_2\to\frac{1}{5}\cdot R_2]{} A_3=M_2\left(\tfrac{1}{5}\right)\cdot A_2=\begin{bmatrix}1&0&0&0\\0&1/5&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\cdot A_2=\begin{bmatrix}1&2&0&1\\0&1&0&1\\0&-3&-1&-1\\0&1&0&1\end{bmatrix}\leftarrow R_1$$

C_1

DO NOT CHANGE (INDUCTIVE STEP).

$$A_4\xrightarrow[\substack{R_3\to R_3+3R_2\\ R_4\to R_4-R_2}]{} T_{32}(3)\cdot T_{42}(-1)\cdot A_3=\begin{bmatrix}1&2&0&1\\0&1&0&1\\0&0&-1&2\\0&0&0&0\end{bmatrix}$$

$$A_5\xrightarrow[R_3\to(-1)R_3]{} M_3(-1)A_4=\begin{bmatrix}1&\textcircled{2}&0&1\\0&1&0&1\\0&0&1&-2\\0&0&0&0\end{bmatrix}$$ ROW ECHELON

$$A_6\xrightarrow[R_1\to R_1+(-2)R_2]{} T_{12}(-2)A_5=\begin{bmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&-2\\0&0&0&0\end{bmatrix}$$ ROW REDUCED ECHELON

SYSTEMS OF LINEAR EQUATIONS.

A system of n Linear equation in m unknows is

$$\begin{cases}a_{11}x_1+a_{12}x_2+\cdots+a_{1m}x_m=b_1\\ \vdots\\ a_{n1}x_1+a_{n2}x_2+\cdots+a_{nm}x_m=b_m\end{cases}$$

$a_{ij}, b_i \in F$. are the coefficients,
$x_1, x_2 \dots x_m$ unknows.

Problem: SOLVE THE SYSTEM: Find all $(x_1, x_2, \dots x_m)\in F^m$ that satisfy the system.

Remark:

WE CAN USE MATRICES AND PRODUCT OF MATRICES TO REPRESENT THE SYSTEM.

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1m}\\ \vdots&&&\\ a_{n1}&a_{n2}&\cdots&a_{nm}\end{bmatrix},\quad b=\begin{bmatrix}b_1\\ \vdots\\ b_m\end{bmatrix},\quad x=\begin{bmatrix}x_1\\x_2\\ \vdots\\ x_m\end{bmatrix}\Rightarrow A\cdot x=b.$$

$S=$ solutions for $(*)$ $=\{(x_1,\dots x_m)\in F^n : Ax=b\}\subseteq F^n$.

Def: A SYSTEM $Ax=b$ IS CALLED:

1) HOMOGENEOUS if $b=0$.

2) INCONSISTENT if $S=\emptyset$.

3) CONSISTENT if $S\neq\phi$

4) CONSISTENT INDEPENDENT if $\#S=1$.

DEPENDENT if $\#S>1$.

Remark:

1) Any Homogeneous system is Consistent since $(x_1, x_2, \cdots x_m)=(0, 0, \cdots 0)$.

2) The set of solutions of any Homogeneous system is a subspace of F^n.

3) If $b\neq 0$, S is not a subspace, since $(0, 0, \cdots 0)\notin S$.

Proof (2):

① $(0, 0, 0 \ldots 0)\in S=\{(x_1 x_2 \ldots x_n)\in F^n : Ax=0\}\subseteq F^n$.

② $Ax_1=0$. $Ax_2=0$. $\Rightarrow A(x_1+x_2)=Ax_1+Ax_2=0$. $\Rightarrow x_1, x_2\in S\Rightarrow x_1+x_2\in S$.

③ $Ax=0$, $\lambda\in F$. $\Rightarrow A\cdot(\lambda x)=\lambda\cdot Ax=\lambda\cdot 0=0$. $x\in S$. $\lambda\in F\Rightarrow\lambda x\in S$.

THEO: Let $S=\{(x_1 \ldots x_m)\in F^m : Ax=b\}$, $S_0=\{(x_1 \ldots x_m)\in F^m : Ax=0\}$, and let $z_0\in S$.

then: $S=\{z_0+w, w\in S_0\}$.

Proof:

$S\supseteq\{z_0+w, w\in S_0\}$: $A(z_0+w)=Az_0+Aw=b+0=b$ ✓.

$S\subseteq\{z_0+w, w\in S_0\}$: $z_1\in S$. $w=z_1-z_0$: $Aw=A(z_1-z_0)=Az_1-Az_0=b-b=0$. $\Rightarrow w=z_1-z_0$.

$\Rightarrow z_1=w+z_0$. ✓

Corollary: If S is a CONSISTENT DEPENDENT SYSTEM, then $\#F\leq\#S$.

In particular, if $\#F=\infty$, then the possibilities of S are: ① $\#S=0$ ② $\#S=1$ ③ $\#S=\infty$

Proof:

If S is CONSISTENT DEPENDENT, take $z_1, z_2\in S$, $z_1\neq z_2$. $\Rightarrow w=z_1-z_2\in S_0$, $w\neq 0$. Since S_0 is a subspace of F^n, then $\{\lambda w, \lambda\in F\}\subseteq S_0$, $\#\{\lambda w, \lambda\in F\}=\#F$. Since $F\longleftrightarrow\{\lambda w, \lambda\in F\}$. Bijection.

$\lambda\longleftrightarrow\lambda w$.

$\{z_0+\lambda w, \lambda\in F\}\subseteq S$, $\#\{z_0+\lambda w, \lambda\in F\}=\#F$. $\Rightarrow\#F\leq\#S$.

THEO. Let $Ax=b$ be a system of LINEAR EQUATIONS. Consider $[A \vdots b]\in M_{m\times(n+1)}(F)$ the matrix associated to the system. If $[A' \vdots b']$ is obtained from $[A \vdots b]$ by applying ROW operations. then:

$S=\{(x_1, \cdots x_m)\in F^m : Ax=b\}=S'=\{(x_1 \ldots x_m)\in F^m : A'x=b'\}$.

Proof:

$$[A \vdots b]\xrightarrow{E_1}\xrightarrow{E_2}\cdots\xrightarrow{E_k}[A' \vdots b']\Leftrightarrow E_k\cdots E_1[A \vdots b]=[A' \vdots b']\Leftrightarrow\begin{cases}E_k\cdots E_1 A=A'.\\ E_k\cdots E_1 b=b'.\end{cases}$$

$S\subseteq S'$: if $Ax=b\Rightarrow A'x=E_k\cdots E_1 Ax=E_k\cdots E_1 b\Rightarrow A'x=b'$. ✓

$$S\supseteq S': \text{if }\begin{cases}E_k\cdots E_1 A=A'\\ E_k\cdots E_1 b=b'\end{cases}\Rightarrow\begin{cases}E_1^{-1}\cdots E_k^{-1}A'=A.\\ E_1^{-1}\cdots E_k^{-1}b'=b\end{cases}\Rightarrow Ax=E_1^{-1}\cdots E_k^{-1}A'x=E_1^{-1}\cdots E_k^{-1}b'=b\Rightarrow Ax=b.$$

$\Rightarrow S=S'$.

LINEAR MAPS.

(F-vector spaces, V,W are F-vector spaces, want to compare them).

Def: A linear map from V to W is a function $T: V \to W$. st:

$T(v_1+v_2) = T(v_1)+T(v_2)$. $\forall v_1, v_2 \in V$.

$T(\lambda v) = \lambda \cdot T(v)$. $\forall \lambda \in F$. $v \in V$.

NOTATION: $\mathcal{L}_F(V,W) = \mathrm{Hom}_F(V,W)$ = SET OF All LINEAR MAPS FROM V to W. For V, W two vector spaces.

Example:

1) $T: V \to W$, $T(v) = 0_W$ is a linear map.

2). $Id: V \to V$ $Id(v) = v$. is ✓.

3) $T: C \to C$ $T(a+bi) = a-bi$ $\Rightarrow$ $T \in \mathrm{Hom}_R(C,C)$. but $T \notin \mathrm{Hom}_C(C,C)$.

C is an R-vector space $\lambda \in R$: $T(\lambda(a+bi)) = T(\lambda a+\lambda bi) = \lambda a - \lambda bi$.

$\lambda T(a+bi) = \lambda a - \lambda bi$ $T \in \mathrm{Hom}_R(V,W)$.

C is an C-vector space. $\lambda \in C$: $\lambda = x+iy$.

$T((x+iy)\cdot(a+bi)) = T((xa-yb)+(xb+ay)i) = xa-yb-(xb+ay)i$.

$(x+yi)\cdot T(a+bi) = (x+yi)(a-bi) = ax+by+(ay-bx)i$. not equal.

So $T \notin \mathrm{Hom}_C(C,C)$.

Example.

1) Describe the set $\mathrm{Hom}_F(F,F)$. $= \{T: F \to F,\ T(a) = aT(1).\ \text{for } T(1) \in F\}$.

$T: F \to F$.

$T(a) = T(a \cdot 1) = a \cdot T(1)$. T is uniquely determined by $T(1)$. $(\mathrm{Hom} \subseteq \{\ \})$.

let $T(1) = c. \in F$.

Define $T(a) = T(a \cdot 1) = a \cdot T(1) = a \cdot c$.

Check: T is a Linear map. $\Rightarrow (\mathrm{Hom} \supseteq \{\ \})$.

$T(a+a') = (a+a') \cdot T(1) = (a+a') \cdot c = ac + a'c. = T(a) + T(a')$

$T(\lambda a) = \lambda a \cdot T(1) = \lambda a \cdot c = \lambda \cdot ac = \lambda \cdot T(a)$.

2) Describe $\mathrm{Hom}_R(C,C) = \{T: C \to C,\ T(a+bi) = a\alpha + b\beta,\ \text{for } \alpha, \beta \in C\}$.

$T: C \to C$. $T(a+bi) = T(a) + T(bi). = T(a \cdot 1) + T(bi) = a\underbrace{T(1)}_{\alpha} + b\underbrace{T(i)}_{\beta}$.

T is uniquely determined by $T(1)$, $T(i)$. $(\mathrm{Hom}_R(C,C) \subseteq \{\ \})$.

Now, check T is linear map.

$T(a+bi) = a\alpha + b \cdot \beta$

$T((a+bi)+(c+di)) = T((a+c)+(b+d)i) = (a+b)\alpha + (b+d)\beta$. $T(a+bi)+T(c+di) = a\alpha + b\beta + c\alpha + d\beta = \ldots$

$T(\lambda(a+bi)) = T(\lambda a + \lambda bi) = \lambda a\alpha + \lambda b\beta. = \lambda T(a+bi)$.

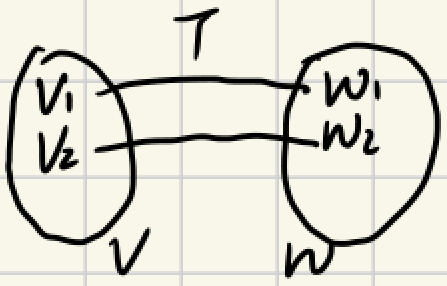
$\Rightarrow (\mathrm{Hom} \supseteq \{\ \})$.

3) Describe the set $\mathrm{Hom}_F(F^2, W) = \{T: F^2 \to W: T(x,y) = x\alpha + y\beta.\ \text{for some } \alpha, \beta \in W\}$.

$T: F^2 \to W$. $T(x,y) = T(x(1,0) + y(0,1)). = xT(1,0) + yT(0,1)$. $\Rightarrow$ T is uniquely determined by $T(1,0)$, $T(0,1)$. Now check it is LM. $\subseteq$.

$T: F^2 \to W$ $T(x,y) = x\alpha + y\beta$. ... ?.

THEO: let $\{v_1 \dots v_n\}$ be basis of V. let $\{w_1, \dots w_n\}$ in W. $\exists ! T : V \to W$ Linear Map st $T(v_i) = w_i$, $\forall i = 1 \dots n$.



Proof:

$\exists T(v) = T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1w_1 + a_2w_2 + \cdots + a_nw_n$

(Well define, since $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. in a unique way). $\Rightarrow T$ is LM. =

$T(v+u) = T((a_1v_1 + \cdots + a_nv_n) + (b_1v_1 + \cdots + b_nv_n)) = T((a_1+b_1)v_1 + (a_2+b_2)v_2 + \cdots + (a_n+b_n)v_n)$.

$= (a_1+b_1)w_1 + (a_2+b_2) + w_2 + \cdots + (a_n+b_n)w_n = (a_1w_1 + a_2w_2 + \cdots + a_nw_n) + (b_1w_1 + b_2w_2 + \cdots + b_nw_n)$

$= T(v) + T(u)$.

$T(\lambda v) = T(\lambda(a_1v_1 + a_2v_2 + \cdots + a_nv_n)) = \lambda a_1w_1 + \lambda a_2w_2 + \cdots + \lambda a_nw_n = \lambda T(v)$.

Proof: $T(v_i) = w_i$.

$T(v_i) = T(0 \cdot v_1 + 0 \cdot v_2 + \cdots + 1 \cdot v_i + \cdots + 0 \cdot v_n) = 0 \cdot w_1 + 0 \cdot w_2 + \cdots + 1 \cdot w_i + \cdots + 0 \cdot w_n = w_i$.

let $T \in Hom_F(V, W)$ st $T(S_i) = w_i$ $\forall i = 1, \dots n$.

$T(v) = T(a_1v_1 + \cdots + a_nv_n) = T(a_1v_1) + T(a_2v_2) + \cdots + T(a_nv_n) = a_1w_1 + a_2w_2 + \cdots + a_nw_n$.

Example:

There is no linear map $T : R^2 \to R^3$ st $T(1,0) = (1,2,3)$, $T(0,1) = (3,2,1)$. $T(1,1) = (0,1,0)$.

If $T \in Hom_R(R^2, R^3)$: $T(1,0) = (1,2,3)$. $T(0,1) = (3,2,1)$. $\Rightarrow T(x,y) = x \cdot T(1,0) + y T(0,1)$.

$= x \cdot (1,2,3) + y(3,2,1) = (x+3y, 2x+2y, 3x+y)$. $\Rightarrow T(1,1) = (4,4,4) \neq (0,1,0)$.

Example.

Find LM $T : R^4 \to R^2$.