3.3 Lecture

Fields

Def: A Field is a non-empty set F with two algebraic structures:

$+ : F \times F \to F$ Addition $(a,b) \to a+b$.

$\cdot : F \times F \to F$ Product $(a,b) \to a \cdot b$.

Satisfying the following axioms:

Associativity: S_1 $(a+b)+c = a+(b+c)$ $\forall a,b,c \in F$.

P_1 $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Commutativity: S_2 $a+b = b+a$ $\forall a,b \in F$

P_2 $a \cdot b = b \cdot a$

Identity: S_3 $\exists\, 0 \in F$: $a+0 = 0+a = a$ $\forall a \in F$.

P_3 $\exists\, 1 \in F$: $a \cdot 1 = 1 \cdot a = a$ $\forall a \in F$.

Inverse: S_4 $\forall a \in F$, $\exists -a \in F$: $a+(-a) = (-a)+a = 0$

P_4 $\forall a \in F$, $a \neq 0$, $\exists\, a^{-1} \in F$: $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

Distributivity $a \cdot (b+c) = a \cdot b + a \cdot c$ $\forall a,b,c \in F$.

Property: Any polynomial equation in $\mathbb{C}$ has a solution in $\mathbb{C}$.

$\mathbb{C} = \mathbb{R}^2 = \{(a,b),\ a,b \in \mathbb{R}\}$

$(a,b)+(c,d) = (a+c,\ b+d)$.

$(a,b) \cdot (c,d) = (ac-bd,\ ad+bc)$.

3.5 Lecture

$\mathbb{C} = \mathbb{R}^2 \longleftrightarrow \mathbb{R}$: $z = (a,b) = (a,0) + (b,0)(0,1)$

$(a,0) \longleftrightarrow a$ $\qquad = a + bi$

Notation : $i = (0,1)$ $\qquad \begin{cases} i^2 = (-1,0) = -1 \\ i^2 + 1 = 0 \end{cases}$

Def: Let $z = (a,b) = a+bi \in \mathbb{C}$

1): MODULE : $|z| = \sqrt{a^2+b^2}$ = Distance $((a,b),(0,0))$.

2): Conjugate : $\bar{z} = a - bi$

Properties about conjugate :

1) $\bar{\bar{z}} = z$
2) $z + \bar{z} = 2\,\mathrm{Re}\,z$
3) $z - \bar{z} = 2\,\mathrm{Im}\,z\, i$
4) $z = \bar{z} \iff z = \mathrm{Re}\,z \in \mathbb{R}$.
5) $\overline{z+w} = \bar{z} + \bar{w}$
6) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

$\forall z, w \in \mathbb{C}$.

Proof (2) $z + \bar{z} = (a+bi) + (a-bi) = 2a = 2\,\mathrm{Re}\,z$.

(5) $\overline{z+w} = \overline{(a+bi)+(c+di)} = \overline{(a+c)+(b+d)i} = (a+c) - (b+d)i$

$\bar{z} + \bar{w} = \overline{a+bi} + \overline{c+di} = a - bi + c - di = (a+c) - (b+d)i$

$\square$

Properties of Module :

1) $z \cdot \bar{z} = |z|^2$
2) $|z| = |-z| = |\bar{z}|$
3) $|z \cdot w| = |z| \cdot |w|$.

$|z+w| \neq |z| + |w|$.

4) : $\frac{|z|}{|w|} = \left|\frac{z}{w}\right|$. $w \neq 0$.

Proof: 1) $z\cdot\bar{z} = (a+bi)(a-bi) = a^2+abi-abi+b^2 = a^2+b^2 = \left(\sqrt{a^2+b^2}\right)^2 = |z|^2$.

3) Lemma: If $x, y \in \mathbb{R}$, $x, y \geq 0$. Then $x = y \iff x^2 = y^2$.

Proof: $x^2 = y^2 \iff x^2 - y^2 = 0 \iff (x-y)(x+y) = 0 \iff x - y = 0$

$x = y$ ☐

By definition, $|z| = \sqrt{a^2+b^2} \in \mathbb{R}$ and $|z| \geq 0$.

By lemma: $|z\cdot w| = |z||w| \iff |z\cdot w|^2 = (|z||w|)^2$

$$|z\cdot w|^2 \underset{(1)}{=} (z\cdot w)\overline{(z\cdot w)} \underset{(6)}{=} (z\cdot w)(\bar{z}\cdot\bar{w}) = (z\cdot\bar{z})(w\cdot\bar{w}) \underset{(1)}{=} |z|^2\cdot|w|^2$$

$$= (|z|\cdot|w|)^2.$$

☐

Remark: $z = a+bi \neq 0 \Rightarrow z^{-1} = \dfrac{a}{a^2+b^2} + \dfrac{-b}{a^2+b^2}i$

$z\cdot\bar{z} = |z|^2 = a^2+b^2 \in \mathbb{R} > 0 \implies z\cdot\bar{z}\cdot\dfrac{1}{a^2+b^2} = 1 \implies z^{-1} = \dfrac{\bar{z}}{a^2+b^2} = \dfrac{a}{a^2+b^2} + \dfrac{-b}{a^2+b^2}i$

☐

(Dot)
Inner product $\langle \underset{z}{(a,b)}, \underset{w}{(c,d)} \rangle = ac + bd$

$\neq$

product $(a,b)(c,d) = (ac-bd,\ ad+bc)$

Lemma: $x, y \in \mathbb{R}$, $x, y \geq 0$, then $x \geq y \iff x^2 \geq y^2$.

Proof: $x^2 \geq y^2 \iff x^2 - y^2 \geq 0 \iff (x+y)(x-y) \geq 0 \iff x - y \geq 0 \iff x \geq y$

☐.

Properties:

1) $\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z|$

2) $|z+w| \leq |z| + |w|$.

Proof: 1): $z = a + bi \Rightarrow a \le |a| = \sqrt{a^2} \le \sqrt{a^2+b^2}$ since $a^2 \le a^2 + b^2$.

$\Rightarrow \operatorname{Re} z \le |\operatorname{Re} z| \le |z|$.

2): $|z+w| \le |z| + |w| \Longleftrightarrow (|z+w|)^2 \le (|z|+|w|)^2$.

$$|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z}+\bar{w}) = z\cdot\bar{z} + z\cdot\bar{w} + w\cdot\bar{z} + w\cdot\bar{w}$$

$$= |z|^2 + z\cdot\bar{w} + \overline{\bar{w}\cdot z} + |w|^2 = |z|^2 + 2\operatorname{Re}(z\cdot\bar{w}) + |w|^2$$

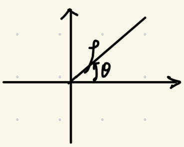
$$\le |z|^2 + 2|z\cdot\bar{w}| + |w|^2 = |z|^2 + 2|z||\bar{w}| + |w|^2$$

$$= |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2.$$

☒.

Polar form

$z = a + bi = \rho_\theta$



z is uniquely defined by (a, b) and it is uniquely defined by (ρ, θ).

有限域的构造:

有限域的元素个数必须是某个素数的幂，即 p^n，其中 p 是素数，n 是正整数

对于 $\mathbb{F}_4$，元素个数是 4，因此 $p=2$, $n=2$.

有限域的构造通常基于一个不可约多项式. 对于 $\mathbb{F}_4$，我们需要一个 2 次不可约多项式，$n=2$.

为什么选择 $\alpha^2 = \alpha + 1$?

在 $\mathbb{F}_2$（即模 2 的域）上，列出所有 2 次多项式，并检查是否可约.

∵ $\mathbb{F}_2$ 通常表示为 $\{0, 1\}$.

在 $\mathbb{F}_2$ 上，多项式系数只能是 0 或 1. 因此：

1. $x^2 = x \cdot x$ 可约
2. $x^2+1 = (x+1)^2$ 可约（在 $\mathbb{F}_2$ 上）.
3. $x^2+x = x(x+1)$ 可约
4. x^2+x+1. 不可约.

$(x+1)^2 = x^2+2x+1$

∵ 在 $\mathbb{F}_2$ 上，$2=0$ ∴ $2x=0$ ∴ $(x+1)^2 = x^2+1$.

为什么要在模2的域($\mathbb{F}_2$)上讨论$\mathbb{F}_4$的构造?

q是元素个数;

对于$\mathbb{F}_4$,元素个数是4,因此$q=4=2^2$,意味着:

- 基域是$\mathbb{F}_2$(因为$p=2$)
- 扩展次数$n=2$.

为什么$a^2=-a-1=a+1$.?

在构造$\mathbb{F}_4$时,我们引入了新元素a,并定义它满足不可约多项式的关系:

$$a^2+a+1=0 \Rightarrow a^2=-a-1$$

关键点: · 在$\mathbb{F}_2$上,$-a=a$ 因为$-1=1$ 所以$-a=1\cdot a=a$.

$$\Rightarrow a^2=a+1.$$

$\mathbb{F}_2$加法表

+	0	1
0	0	1
1	1	0

3.10 Lecture

Roots:

We want to find all $z \in \mathbb{C}$ that satisfy the equation $z^n = w$, for some $w \in \mathbb{C}$, $n \in \mathbb{N}$.

If $w = 0$, then $z^n = 0 \iff |z|^n = |z^n| = 0 \iff |z| = 0 \iff z = 0$.

Theorem: Let $w \in \mathbb{C}$, $w \neq 0$, $n \in \mathbb{N}$.

then $z^n = w \iff z = \sqrt[n]{|w|}\left(\cos\frac{\operatorname{Arg} w + 2k\pi}{n} + \sin\frac{\operatorname{Arg} w + 2k\pi}{n}\, i\right)$, $k = 0, 1, 2, \cdots, n-1$

Polynomials on one variable x with coefficients on a Field F.

Formal expression $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_i \in F$, $n \in \mathbb{N}$

Definition: $f(x) = a_0 + a_1 x + \cdots + a_n x^n = g(x) = b_0 + b_1 x + \cdots + b_n x^n$.

$\iff a_k = b_k$, $\forall k \geq 0$.

3.12 Lecture

In $\mathbb{Z}$, any $n \in \mathbb{Z}$, $n \neq 0, 1, -1$, can be written in a unique way as (± 1) times a product of positive prime numbers

p prime $\Leftrightarrow$ $p = a \cdot b$ $a, b \in \mathbb{Z}$.
$\Rightarrow a = \pm 1$ or $b = \pm 1$.

$(\mathbb{Z}, +, \cdot)$ Ring $\leftarrow$ Positive Prime numbers

$(F[x], +, \cdot)$ Ring $\leftarrow$ Monic Irreducible Polynomials.

Def: A polynomial $f(x) \in F[x]$ is called irreducible if $f(x) \neq 0$, $\deg f(x) \neq 0$ and if $f(x) = g(x) \cdot h(x)$ then $\deg g(x) = 0$ or $\deg h(x) = 0$.

Remark: $f(x)$ is irreducible $\Longleftrightarrow$ $f(x) \neq 0$ and it cannot be written as the product of two non-constant polynomials.

Division Algorithm in $F(x)$.

For any $f(x), g(x) \in F(x)$. $g(x) \neq 0$, there exist unique $q(x), r(x) \in F(x)$. such that.

$f(x) = g(x) \cdot q(x) + r(x)$, $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Proof: $X = \{ f(x) - g(x) \cdot h(x),\ h(x) \in F(x) \}$. If $0 \in X \Rightarrow 0 = f(x) - g(x) \cdot h(x) \Rightarrow r(x) = 0$ ✓

If $0 \notin X$, $\emptyset \neq \deg(X) = \{ \deg(f(x) - g(x) \cdot h(x)),\ h(x) \in F(x) \} \subseteq \mathbb{N}_0$

By W.O.P. $\exists\, s =$ First element in $\deg(X)$, $s = \deg(\underbrace{f(x) - g(x) \cdot h(x)}_{r(x)})$

Assume $\deg(r(x)) > \deg(g(x))$ $\rightarrow$ Contradiction.
Prove. since s is first element.

Take $h(x) = \frac{f(x)}{g(x)} - 1$, then $g(x)$ is in the set. $\deg(f(x) - g(x) \cdot h(x)) \overset{\boxtimes}{>} \deg(g(x))$.

Def: Let $f(x) \in F(x)$, $a \in F$, we say that a is a root of $f(x)$ if $f(a)=0$.

Remainder theorem:

let $f(x) \in F(x)$, $a \in F$, then the remainder of the division of $f(x)$ by $x-a$ is $f(a)$.

Proof: By the division algorithm, $\exists!\ q(x), r(x) \in F(x)$: $f(x) = (x-a)\cdot q(x) + r(x)$, with $r(x)=0$ or $\deg r(x) < \deg((x-a)) = 1 \Rightarrow \deg r(x) = 0 \Rightarrow r(x) = C$ constant.

$f(x) = (x-a)\cdot q(x) + C \quad f(a) = 0\cdot q(a) + C \Rightarrow C = f(a)$. ☑

Theorem:

$f(x) \in \mathbb{R}(x)$, $f(x) \neq 0$. $z = a+bi$ is a root $\Longleftrightarrow \bar{z} = a-bi$ is a root.

Proof: $\Rightarrow$): $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_i \in \mathbb{R}$.

If $z = a+bi$ is a root $\Rightarrow 0 = f(z) = a_0 + a_1 z + \cdots + a_n z^n$

$$f(\bar{z}) = a_0 + a_1\bar{z} + \cdots + a_n \bar{z}^n$$
$$= \bar{a_0} + \bar{a_1}\bar{z} + \cdots + \bar{a_n}\bar{z}^n$$
$$= \overline{a_0 + a_1 z + \cdots + a_n z^n} = \bar{0} = 0.$$

$\Leftarrow$) $f(\bar{z}) = 0$, we have prove $f(\bar{\bar{z}}) = 0$, $\bar{\bar{z}} = z$. ☑

Theorem: Let $f(x) \in \mathbb{Z}[x]$, $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $\deg f(x) > 0$.

If $\frac{a}{b} \in \mathbb{Q}$, $\gcd(a,b) = 1$, $f(\frac{a}{b}) = 0$, then $a \mid a_0$, $b \mid a_n$.

Proof: $0 = f(\frac{a}{b}) = a_0 + a_1 \cdot \frac{a}{b} + \cdots + a_n (\frac{a}{b})^n$

$$0 = 0\cdot b^n = a_0 b^n + a_1 a b^{n-1} + \cdots + a_n \cdot a^n \in \mathbb{Z}.$$

$a_0 b^n = a\cdot x \quad x \in \mathbb{Z} \Rightarrow a \mid a_0 b^n$, $\gcd(a,b)=1 \Rightarrow a \mid a_0$

$a_n \cdot a^n = b\cdot y$, $y \in \mathbb{Z} \Rightarrow b \mid a_n \cdot a^n$. $\gcd(a,b) = 1 \Rightarrow b \mid a_n$. ☑

3.17 Lecture.

Vector space:

Def: Let F be a field, V a set. We say that V is an F - vector space if there are two operations:

- Addition of vectors: $+ : V \times V \to V$
  $(v, w) \to v+w$

- Product of scalars: $\cdot : F \times V \to V$
  $(\lambda, v) \to \lambda \cdot v$.

Properties: Let V be an F-vector space, $v, w, u \in V$, $\lambda \in F$.

1): $v + u = w + u \Rightarrow v = w$.
$u + v = u + w \Rightarrow v = w$

2): $0_F \cdot v = 0_V$, $\lambda \cdot 0_V = 0_V$

3): $\lambda \cdot v = 0_V \Rightarrow \lambda = 0_F$ or $v = 0_V$

4): $-v = (-1_F) \cdot v$.

Proof: 1): $v + u = w + u \xRightarrow[-u \in V]{S_4} (v+u) + (-u) \xRightarrow{S_1} v + (u + (-u)) = w + (u + (-u)) \xRightarrow{S_4} v + 0_V = w + 0_V$
$\xRightarrow{S_3} v = w$

2): $0_F \cdot v + 0_V \underset{S_3}{=} 0_F \cdot v \underset{S_3}{=} (0_F + 0_F) \cdot v \underset{M_1}{=} 0_F \cdot v + 0_F \cdot v \underset{(1)}{\Longrightarrow} 0_V = 0_F \cdot v$.

$\lambda 0_V + 0_V \underset{S_3}{=} \lambda \cdot 0_V \underset{S_3}{=} \lambda (0_V + 0_V) \underset{M_3}{=} \lambda \cdot 0_V + \lambda \cdot 0_V \underset{(1)}{\Longrightarrow} 0_V = \lambda \cdot 0_V$

3): If $\lambda = 0_F$, we are done
If $\lambda \neq 0_F$, $\exists \lambda^{-1} \in F$, $0_V \underset{(2)}{=} \lambda^{-1} \cdot 0_V = \lambda^{-1} \cdot (\lambda \cdot v) \underset{M_2}{=} (\lambda^{-1} \cdot \lambda) v = 1 \cdot v = v$. ✓

4): $v + (-1) \cdot v \underset{M_4}{=} 1 \cdot v + (-1) \cdot v \underset{M_1}{=} (1 + (-1)) \cdot v = 0_F \cdot v \underset{(2)}{=} 0_V$.

(S1) ASSOCIATIVITY: $(v+w)+u = v+(w+u)$, $\forall v,w,u \in V$

(S2) COMMUTATIVITY: $v+w = w+v$, $\forall v,w \in V$

(S3) IDENTITY: $\exists\, 0 \in V$: $v+0 = v = 0+v$, $\forall v \in V$

(S4) INVERSE: $\forall v \in V$, $\exists\, -v \in V$: $v+(-v) = 0 = (-v)+v$

(M1) $(\lambda+\mu)v = \lambda \cdot v + \mu v$, $\forall \lambda,\mu \in F$, $v \in V$

(M2) $(\lambda \cdot \mu)\cdot v = \lambda \cdot (\mu \cdot v)$, $\forall \lambda,\mu \in F$, $v \in V$

(M3) $\lambda \cdot (v+w) = \lambda v + \lambda w$, $\forall \lambda \in F$, $v,w \in V$

(M4) $1 \cdot v = v$, $\forall v \in V$

More properties:

1): 0_V is unique.

2): For any $v \in V$, $-v$ is unique.

Proof: 1) If 0 and $0'$ are identities in V satisfying S_3, then

$$0 = 0 + 0' = 0'$$

2) If v_1 and v_2 are inverses for v then

$$v_1 = v_1 + 0_V \underset{\downarrow}{=} v_1 + (v + v_2) \underset{S_1}{=} (v_1 + v) + v_2 = 0_V + v_2 = v_2.$$

v_2 is the inverse of v

▨

Subspace:

Def: Let V be an F-vector space. A subset $S \subseteq V$ is called a subspace if S is itself an F-vector space with the same operation as V.

Theorem: Let V be an F-vector space, $S \subseteq V$.

1): S is a subspace of V

2): i): $S \neq \emptyset$ ii): For any $v, w \in S \Rightarrow v + w \in S$ iii): For any $v, w \in S \Rightarrow v \cdot w \in S$.

3): a) $0 \in S$ b) = ii) c) = iii).

Theorem: $\emptyset \neq W \subseteq V$, V is vector space

W is subspace $\Longleftrightarrow$ W is closed under addition and mult. by scalers.

3.19 Lecture.

Subspace: $S \subseteq V$. S is an F-vector space with the same operations as V.

How do we check if S is an F-vector space?

If $S \times S \xrightarrow{+} S$, $F \times S \xrightarrow{\cdot} S$.

Theorem: Let V be an F-vector space, let $S \subseteq V$. The following are equivalent:

a): S is a subspace (= S is an F-vector space with the operations inherit from V).

b): i) $S \neq \emptyset$

ii) If $v, w \in S \Rightarrow v + w \in S$.

iii) If $\lambda \in F$, $v \in S \Rightarrow \lambda \cdot v \in S$.

(a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (a) ?

c): i) $0_V \in S$

ii) $v, w \in S \Rightarrow v + w \in S$

iii) $\lambda \in F$, $v \in S \Rightarrow \lambda \cdot v \in S$.

Proof: (a) $\Rightarrow$ (b).

i) We know that any vector space is non-empty then S F-vector space $\Rightarrow S \neq \emptyset$.

ii) iii) If S is a F-vector space with the operations in V, then the restritions:

$S \times S \xrightarrow{+} S$, $F \times S \xrightarrow{\cdot} S$, that is, $v, w \in S \Rightarrow v + w \in S$, $\lambda \in F$, $v \in V \Rightarrow \lambda \cdot v \in S$.

(b) $\Rightarrow$ (c). i) $\Rightarrow$ i')

$S \neq \emptyset \Rightarrow \exists N \in S$. By iii), $0_F \in F$, $v \in S \Rightarrow 0_V = 0_F \cdot v \in S$.
(proved last Lecture)

(c) $\Rightarrow$ (a).

By (c). we know that S admits an addition and a product (Inherit from V).

(c) $\Rightarrow$ (a)

BY (c) WE KNOW THAT S ADMITS AN ADDITION AND A PRODUCT (INHERIT FROM V)

$(F,+,\cdot)$, $S \times S \xrightarrow{+} S$, $F \times S \xrightarrow{\cdot} S$

(S1) $(v+w)+u = v+(w+u)$ $\forall v,w,u \in S$ THIS HOLDS SINCE IT HOLDS $\forall v,w,u \in V$

(S2) WE KNOW $v+w = w+v$ $\forall v,w \in V$. So IN PARTICULAR, $v+w = w+v$ $\forall v,w \in S$.

(S3) = (i') $0_V \in S$

(S4) $v \in S$, WE KNOW THAT $-v = (-1)v$, BY (iii) $-1 \in F$, $v \in S \Rightarrow (-1)v \in S$

M1) $\lambda(u+v) = \lambda u + \lambda v$ $\forall \lambda \in F$, $u,v \in S$

M2) $(\lambda+\mu)v = \lambda v + \mu v$, $\forall \lambda,\mu \in F$, $v \in S$

M3) $(\lambda \cdot \mu) v = \lambda(\mu \cdot v)$, $\forall \lambda,\mu \in F$, $v \in S$

M4) $1 v = v$ $\forall v \in S$

$\leftarrow$ THESE HOLD SINCE THEY HOLD IN V.

Example:

$\mathbb{C}$ is a $\mathbb{C}$-vector space, $\mathbb{R} \subseteq \mathbb{C}$, $\mathbb{R}$ is not a subspace of the $\mathbb{C}$-vector space $\mathbb{C}$ (i) $0 \in \mathbb{R}$. (ii) $x, y \in \mathbb{R} \Rightarrow (x+0i)+(y+0i) = x+y \in \mathbb{R}$.

(iii): $\lambda \in \mathbb{C}$, $x \in \mathbb{R} \overset{?}{\Longrightarrow} \lambda x \in \mathbb{R}$ not true.

$\mathbb{C}$ is an $\mathbb{R}$-vector space.

$\mathbb{R} \subseteq \mathbb{C}$, $\mathbb{R}$ is an subspace

i) $0 = 0 + 0i \in \mathbb{R}$.

ii) $x, y \in \mathbb{R} \Rightarrow x+y = x+0i + y+0i = (x+y) + 0i \in \mathbb{R}$.

iii) $\lambda \in \mathbb{R}$, $x \in \mathbb{R} \Rightarrow \lambda \cdot x = \lambda(x+0i) = \lambda x + 0i \in \mathbb{R}$.

Theorem:

1): If S_1, S_2 are subspaces of a F-vector space $V \Rightarrow S_1 \cap S_2$ is a subspace of V.

2): If $\{S_i, i \in I\}$ " " " " " $\Rightarrow \bigcap_{i \in I} S_i$ is a subspace of V.

Proof: 1): $0 \in S_1, 0 \in S_2 \Rightarrow 0 \in S_1 \cap S_2$ ✓

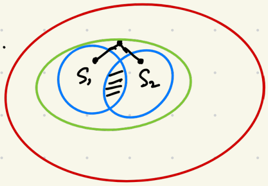
If $v, w \in S_1 \cap S_2 \Rightarrow \begin{cases} v, w \in S_1 \Rightarrow v+w \in S_1 \\ v, w \in S_2 \Rightarrow v+w \in S_2 \end{cases} \Rightarrow v+w \in S_1 \cap S_2$.

If $\lambda \in F$, $v \in S_1 \cap S_2 \Rightarrow \begin{cases} \lambda \in F, v \in S_1 \Rightarrow \lambda v \in S_1 \\ \lambda \in F, v \in S_2 \Rightarrow \lambda v \in S_2 \end{cases}$ $\lambda v \in S_1 \cap S_2$,

▨.

Def: 1): Let S_1, S_2 be subspaces of an F-vector space V

Then $S_1 + S_2$ is the smallest subspace of V containing $S_1 \cup S_2$.



$S_1 = \{(x, y, 0) : x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$

$S_2 = \{(x, 0, z) : x, z \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

$(x,y,0) \in S_1 \subseteq S_1 + S_2 \subseteq \mathbb{R}^3$

$(x,0,z) \in S_2 \subseteq S_1 + S_2 \subseteq \mathbb{R}^3$

$(x,y,0), (x,0,z) \in S_1 + S_2$, $S_1 + S_2$ is a subspace $\Rightarrow (x,y,z) = (x,y,0) + (x,0,z) \in S_1 + S_2$

$\Rightarrow \mathbb{R}^3 \subseteq S_1 + S_2 \subseteq \mathbb{R}^3 \Rightarrow S_1 + S_2 = \mathbb{R}^3$.

Theorem: 1) $S_1 + S_2 = \{v_1 + v_2,\ v_1 \in S_1,\ v_2 \in S_2\}$

2) $S_1 + S_2 + \cdots + S_n = \{v_1 + v_2 + \cdots + v_n,\ v_1 \in S_1, \cdots, v_n \in S_n\}$.

Proof: We have to prove that $S_1 + S_2$ is the smallest (c) subspace (a) contains S_1 and S_2 (b)

(a). i): $0 = \underset{\in S_1}{0} + \underset{\in S_2}{0} \in S_1 + S_2$ since S_1, S_2 are subspaces

ii): $v_1 + v_2,\ w_1 + w_2,\ v_1, w_1 \in S_1,\ v_2, w_2 \in S_2 \Rightarrow (v_1 + v_2) + (w_1 + w_2) = (v_1 + w_1) + (v_2 + w_2) \in S_1 + S_2$

iii): $\lambda \in F,\ v_1 + v_2,\ v_1 \in S_1,\ v_2 \in S \Rightarrow \lambda(v_1 + v_2) = \lambda \cdot v_1 + \lambda \cdot v_2 \in S_1 + S_2$.

(b). $S_1 \subseteq S_1 + S_2$ since $v \in S_1 \Rightarrow v = \underset{\in S_1}{v} + \underset{\in S_2}{0}$

$S_2 \subseteq S_1 + S_2$ since $w \in S_2 \Rightarrow w = \underset{\in S_1}{0} + \underset{\in S_2}{w}$.

(c). We have to prove $S_1 + S_2$ is the smallest one.

Let U be a subspace containing $S_1 \cup S_2$, we have to prove that $S_1 + S_2 \subseteq U$.

$v_1 + v_2 \in S_1 + S_2 \Rightarrow \begin{cases} v_1 \in S_1 \subseteq U \\ v_2 \in S_2 \in U \end{cases} \Rightarrow v_1, v_2 \in U$, U is a subspace $\Rightarrow v_1 + v_2 \in U$.

☒

3.24. Lecture.

Def: 1) A sum of two subspaces. $S_1 + S_2$, is called a <u>direct sum</u> if any vector in $S_1 + S_2$ can be written in a unique way as $v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2$.

That is: $w = v_1 + v_2 = v_1' + v_2'$. $v_1, v_1' \in S_1$, $v_2, v_2' \in S_2$ $\Rightarrow \begin{cases} v_1 = v_1' \\ v_2 = v_2' \end{cases}$.

We denote it as $S_1 \oplus S_2$.

Example: $\mathbb{R}^3 = S_1 + S_2$ Not direct

$= S_1 \oplus S_3 = \{(x, y, 0),\ x, y \in \mathbb{R}\} \oplus \{(0, 0, z),\ z \in \mathbb{R}\}$.

$= T_1 \oplus T_2 \oplus T_3 = \{(x, 0, 0),\ x \in \mathbb{R}\} + \{(0, y, 0),\ y \in \mathbb{R}\} + \{(0, 0, z),\ z \in \mathbb{R}\}$.

$(x, y, z) = (a, 0, 0) + (0, b, 0) + (0, 0, c) \iff \begin{cases} x = a \\ y = b \\ z = c \end{cases}$ Unique way for writing the vector as a sum.

$= T_1 \oplus T_3 \oplus T_4 = \{(x, 0, 0),\ x \in \mathbb{R}\} + \{(0, y, y),\ y \in \mathbb{R}\} + \{(z, 0, z),\ z \in \mathbb{R}\}$.

$(x, y, z) = (a, 0, 0) + (0, b, b) + (c, 0, c) \iff \begin{cases} x = a + c \\ y = b \\ z = b + c \end{cases} \iff \begin{cases} b = y \\ c = z - y \\ a = x - z + y \end{cases}$

Theorem: 1) $S_1 \oplus S_2$ is a direct sum $\iff$ $0 = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2 \Rightarrow v_1 = 0 \wedge v_2 = 0$

that is, 0 can be written in a unique way. As sum of $v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2$.

2) $S_1 \oplus S_2 \oplus \cdots \oplus S_n$ is a direct sum $\iff$ $0 = v_1 + v_2 + \cdots + v_n$, $v_i \in S_i$, $i = 1 \cdots n$
$\Rightarrow v_1 = v_2 = \cdots = v_n = 0$.

Proof: 1) $\Rightarrow$) We know that any $u \in S_1 + S_2$ can be written in a unique way as $u = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2$. In particular. we know that $0 = 0_{\in S_1} + 0_{\in S_2}$ in a unique way then $0 = v_1 + v_2 \Rightarrow v_1 = 0$ and $v_2 = 0$

$\Leftarrow$): Assume $w = v_1 + v_2 = v_1' + v_2'$, $v_1, v_1' \in S_1$, $v_2, v_2' \in S_2$. we want to prove $v_1 = v_1'$, $v_2 = v_2'$.

$$0 = w - w = \underbrace{(v_1 - v_1')}_{\in S_1} + \underbrace{(v_2 - v_2')}_{\in S_2}.$$

By hypothesis, $\begin{cases} v_1 - v_1' = 0 \\ v_2 - v_2' = 0 \end{cases} \Rightarrow \begin{cases} v_1 = v_1' \\ v_2 = v_2' \end{cases}$

$\blacksquare$

Corollary: $S_1 + S_2 = S_1 \oplus S_2$ is a direct sum $\Leftrightarrow S_1 \cap S_2 = \{0\}$.

Proof: We know that $S_1 + S_2 = S_1 \oplus S_2 \Leftrightarrow 0 = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2 \Rightarrow v_1 = 0, v_2 = 0$.

Let's see that $0 = v_1 + v_2$, $v_1 = 0 = v_2 \Longleftrightarrow S_1 \cap S_2 = \{0\}$.

$\Rightarrow$): $v \in S_1 \cap S_2 \Rightarrow -v \in S_1 \cap S_2$ since $S_1 \cap S_2$ is a subspace and $-v = (-1) \cdot v$.

$$0 = \underbrace{v}_{\substack{\in S_1 \cap S_2 \\ \subseteq S_1}} + \underbrace{(-v)}_{\substack{\in S_1 \cap S_2 \\ \subseteq S_2}} \Rightarrow \begin{cases} v = 0 \\ -v = 0 \end{cases} \Rightarrow \{0\} \subseteq S_1 \cap S_2 \subseteq \{0\}$$

$$\Rightarrow S_1 \cap S_2 = \{0\}.$$

$\Leftarrow$): Let $0 = v_1 + v_2$, $v_1 \in S_1$, $v_2 \in S_2 \Rightarrow v_1 = -v_2 \in S_1 \cap S_2 = \{0\}$

$$\Rightarrow \begin{cases} v_1 = 0 \\ v_2 = 0 \end{cases}$$

$\blacksquare$

Remark: This corollary is not true for $n \geq 3$

$$S_1 + S_2 + S_3 = S_1 \oplus S_2 \oplus S_3 \underset{\text{Algebra B}}{\Longleftrightarrow} S_1 \cap (S_2 + S_3) = \{0\}. \quad S_2 \cap (S_1 + S_3) = \{0\}$$
$$S_3 \cap (S_1 + S_2) = \{0\}.$$

Def: Let V be an F-vector space, $v_1, v_2, \ldots, w \in V$.

We say that w is a <u>Linear Combination</u> (L.C.) of $v_1, v_2, \ldots, v_n$ If $\exists \lambda_1, \lambda_2, \ldots, \lambda_n \in F$ such that $w = \lambda_1 \cdot v_1 + \lambda_2 \cdot v_2 + \cdots + \lambda_n \cdot v_n$.

Def: Let V be an F-vector space, $v_1, \cdots, v_n \in V$

$\text{span}(v_1, v_2, \cdots, v_n) = \text{span}(\{v_1, v_2, \cdots, v_n\})$. is the smallest subspace of V containing the set $\{v_1, \cdots, v_n\}$.

Remark: Let $U = \text{span}(v_1, v_2, \cdots, v_n)$.

U is a subspace and $v_i \in U$ for any $i = 1, \cdots, n$.

$$v_i \in U \Longleftrightarrow S_i = \{\lambda v_i, \lambda \in F\} \subseteq U.$$

$\Rightarrow$) $v_i \in U \Rightarrow \lambda \cdot v_i \in U$ since U is a subspace.

$\Leftarrow$) $v_i = 1 \cdot v_i \in U$.

Theorem: $\text{span}(v_1, v_2, \cdots, v_n)$ = smallest subspace of V containing $S_1 \cup S_2 \cup \cdots \cup S_n$ $\quad S_i = \{\lambda v_i, \lambda \in F\}$.

$= S_1 + S_2 + \cdots + S_n$.

$= \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \lambda_i \in F\}$ = set of L.C. of $v_1, \cdots, v_n$

$\mathbb{C} \times \mathbb{C} = \mathbb{C}^2 = \{(a+bi, c+di), a, b, c, d \in \mathbb{R}\}$.

As a $\mathbb{C}$-vector space $\quad \text{span}(\{(1,0), (0,1)\})$.

$(a+bi, c+di) = (a+bi)(1,0) + (c+di)(0,1)$

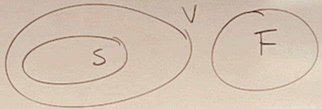
As an $\mathbb{R}$-vector space $\quad \text{span}(\{(1,0), (i,0), (0,1), (0,i)\})$.

$(a+bi, c+di) = a(1,0) + b(i,0) + c(0,1) + d(0,i)$

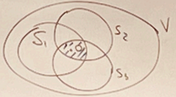
3.26 Lecture.

VECTOR SPACE: $(F,+,\cdot)$ FIELD; $(V,+)$; $F\times V \to V$ + AXIOMS

$\mathbb{C}$-VECTOR SPACE $\mathbb{C}^2$ $\neq$ $\mathbb{R}$-VECTOR SPACE $\mathbb{C}^2$



F-SUBSPACES = SUBSETS OF F-VECTOR SPACES WHICH ARE F-VECTOR SPACES



INTERSECTION OF SUBSPACES IS A SUBSPACES

UNION " " IS NOT ALWAYS A SUBSPACE

SUM OF SUBSPACES: $S_1+S_2+\dots+S_n \overset{DEF}{=}$ SMALLEST SUBSPACE OF V CONTAINING $S_1\cup S_2\cup\dots\cup S_n$

$\underset{THEO}{=} \{v_1+v_2+\dots+v_n,\ v_i\in S_i,\ i=1,\dots,n\}$.

IN PARTICULAR: $S_i=\{\alpha v_i,\ \alpha\in F\}$ (CHECK THAT S_i IS A SUBSPACE OF V).

$S_1+S_2+\dots+S_n=\{\alpha_1 v_1+\alpha_2 v_2+\dots+\alpha_n v_n,\ \alpha_1,\dots,\alpha_n\in F\}$

$\underset{DEF}{=}$ SET OF L.C. OF $v_1,v_2,\dots,v_n$.

$\underset{DEF}{=}$ SMALLEST SUBSPACE OF V CONTAINING $S_1\cup S_2\cup\dots\cup S_n$

$\underset{PROPOSITION}{=}$ " " " " $\{v_1,v_2,\dots,v_n\}$

PROPOSITION: U IS A VECTOR SPACE

$v\in U \iff \{\alpha u,\ \alpha\in F\}\subseteq U$.

EXAMPLES:

1) $\mathbb{C}$ - VECTOR SPACE $\mathbb{C}^2 \Longrightarrow \mathbb{C}^2 = \text{SPAN}\left((1,0),(0,1)\right)$

$$(a+bi, c+di) = (a+bi)(1,0) + (c+di)\cdot(0,1)$$

$\{(1,0),(0,1)\}$ DOES NOT SPAN THE $\mathbb{R}$-VECTOR SPACE $\mathbb{C}^2$.

$$(i, 4\pi) \neq \mu_1(1,0) + \mu_2(0,1) = (\mu_1, \mu_2), \quad \mu_1, \mu_2 \in \mathbb{R}$$

$\mathbb{R}$ - VECTOR SPACE $\mathbb{C}^2 \Longrightarrow \mathbb{C}^2 = \text{SPAN}\left((1,0),(i,0),(0,1),(0,i)\right\}$

$$(a+bi, c+di) = a(1,0) + b(i,0) + c(0,1) + d(0,i), \quad a,b,c,d \in \mathbb{R}$$

2) LET $S = \{(x+y, x, 2y) : x,y \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

CHECK THAT S IS A SUBSPACE OF THE $\mathbb{R}$-VECTOR SPACE $\mathbb{R}^3$.

FIND A SPANNING SET FOR S

$$(x+y, x, 2y) = (x, x, 0) + (y, 0, 2y) = x(1,1,0) + y(1,0,2)$$

$$S = \text{SPAN}\big((1,1,0), (1,0,2)\big).$$

3) $\text{SPAN}((1,2,0),(0,1,1)) \underset{\text{DEF}}{=}$ SMALLEST SUBSPACE OF $\mathbb{R}^3$ CONTAINING $\{(1,2,0),(0,1,1)\}$

$= \{a(1,2,0)+b(0,1,1),\ a,b\in\mathbb{R}\}$

$= \{(a,2a+b,b),\ a,b\in\mathbb{R}\} \not\ni (2,3,10)$ with $a = 2$, $b = 10$: $3 \neq 2a+b = 22+10$

$= \{(x,y,z) : y = 2x+z,\ x,z\in\mathbb{R}\}$

$= \{(x,y,y-2x),\ x,y\in\mathbb{R}\}$ $\ni (2,14,10)$, with $2 = x$, $14 = y$, $10 = y-2x$

$= \text{SPAN}((1,0,-2),(0,1,1))$

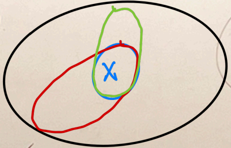
$(x,y,y-2x) = (x,0,-2x) + (0,y,y)$

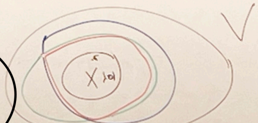
$= x(1,0,-2) + y(0,1,1)$

Theo: Let V be an F-vector space, $X \subseteq V$ a subset.

Consider $\mathcal{F} = \{ S \subseteq V,\ S \text{ a subspace of } V,\ X \subseteq S \}$

Then $\operatorname{span}(X) = \bigcap_{S \in \mathcal{F}} S$





Proof: $\mathcal{F} \neq \emptyset$ since $V \in \mathcal{F}$. $\bigcap_{S \in \mathcal{F}} S$ is a subspace of V ✓

$X \subseteq \bigcap_{S \in \mathcal{F}} S$ ✓ since $X \subseteq S \ \forall S \in \mathcal{F}$

$\operatorname{Span}(X) = \bigcap_{S \in \mathcal{F}} S \Longleftarrow$ smallest $\quad \bigcap S \subseteq S \ \forall S \in \mathcal{F}$.

$X = S_1 \cup S_2 \cup \dots \cup S_n$

or $X = \{v_1, v_2, \dots, v_n\}$

$\operatorname{Span}(X)$ = smallest subspace of V containing X

||

L.C. of vectors in X.

Finite dimensional vector space

Def: An F - vector space V is finite dimensional if $\exists X \subseteq V$, X finite, s.t.

$$V = span(X).$$

that is. $X = \{v_1, v_2, \cdots, v_n\}$, $V = span(v_1, v_2, \cdots, v_n) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \lambda_i \in F\}$

Notation: V has finite dimension: $\dim_F V < \infty$

V has infinite dimension: $\dim_F V = \infty$.

Examples: 1) $\dim_F F^n < \infty$ since $(x_1, x_2, \cdots, x_n) = x_1 \underbrace{(1, 0, \cdots, 0)}_{e_1} + x_2 \underbrace{(0, 1, 0, \cdots, 0)}_{e_2} + \cdots + x_n \underbrace{(0, \cdots, 1)}_{e_n}$

$$\Rightarrow F^n = span(e_1, e_2, \cdots, e_n).$$

2). $\dim_F F[x] = \infty$

Assume $\dim_F F[x] < \infty$. Then $\exists X = \{p_1(x), \cdots, p_s(x)\} \subseteq F[x]$.

such that $F[x] = span(X) = span(p_1(x), p_2(x), \cdots, p_s(x))$.

$$= \{\lambda_1 p_1(x) + \lambda_2 p_2(x) + \cdots + \lambda_s p_s(x), \lambda_i \in F\}.$$

If $m = \max\{\deg p_i(x), i = 1, \cdots, s\}$

$q(x) = \lambda_1 p_1(x) + \lambda_2 p_2(x) + \cdots + \lambda_s p_s(x)$.

$q(x) = 0$ or $\deg(q(x)) \leq m$

$\Rightarrow x^{m+1} \notin span(p_1(x), \cdots, p_s(x)) = F[x]$. Contradiction!.

REMARK: WHEN WE SAY SEQUENCE OR LIST OF VECTORS, WE ALLOW TO REPEAT VECTORS.

SET $\{v_1, v_2, v_3\}$ THIS MEANS $v_1 \neq v_2, v_1 \neq v_3, v_2 \neq v_3$. , $\{v_1, v_2, v_3\} = \{v_1, v_3, v_2\}$.

SEQUENCE $(v_1, v_1, v_2, v_3) \neq (v_1, v_2, v_3)$

$\neq (v_1, v_2, v_1, v_3)$.

SET: NO REPETITION, NO ORDER

Def: A sequence of vectors $(v_1, v_2, \cdots, v_n)$ in an F-vector space V is called linear dependent L.D.

If $\exists\ \lambda_1, \lambda_2, \cdots, \lambda_n \in F$, $(\lambda_1, \lambda_2, \cdots, \lambda_n) \neq (0, 0, \cdots, 0)$ s.t.

$$\lambda_1 V_1 + \lambda_2 V_2 + \cdots + \lambda_n V_n = 0.$$

Independence:

Vectors $x_1, x_2, \cdots, x_n$ are independent if no combination gives zero vector

(Not all c are 0) $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \neq 0$. 如果ci零向量，那一定dependent.

Conversion: $\emptyset$ is L.I.

3.31 Lecture

Remarks:
1) $\emptyset$ is L.I.
2) $\{v, v \in V, v \neq 0\}$ L.I. $\quad 0 = \lambda \cdot v \Rightarrow \lambda = 0$.
3) $(v, v, v_3, v_4, \cdots)$ L.D $\quad 0 = 1 \cdot v + (-1)v + 0 v_3 + 0 v_4 = 1v + (-1)v$
4) $(0, v_2, v_3, \cdots)$ L.D. $\quad 0 = 1 \cdot 0$.
5) (v, w) L.I. $\Leftrightarrow$ $v, w \neq 0$ and $w \neq \lambda v$, $\lambda \in F$ $\Leftrightarrow$ $v, w \neq 0$ and $w \notin \text{span}(v)$.

Proposition: Let V be an F-vector space, $\{v_1, \cdots, v_n\} \subseteq V$. then the list $(v_1, v_2, \cdots, v_n)$ is L.D $\Leftrightarrow$ $\exists j$: $v_j \in \text{span}(v_1, v_2, \cdots, v_{j-1})$, for some $1 \leq j \leq n$

In this case: $\text{span}(v_1, v_2, \cdots, v_n) = \text{span}(v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_n)$.

Proof: $\Rightarrow$) Assume $(v_1, v_2, \cdots, v_n)$ is L.D $\Rightarrow$ $0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$, $\lambda_i \in F$ s.t. $\{k : \lambda_k \neq 0\}$ is non-empty and finite. Let $j = \max\{k : \lambda_k \neq 0\}$, then:

$$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1} + \lambda_j v_j$$

$$\Rightarrow v_j = \left(\frac{-\lambda_1}{\lambda_j}\right) v_1 + \left(\frac{-\lambda_2}{\lambda_j}\right) v_2 + \cdots + \left(\frac{-\lambda_{j-1}}{\lambda_j}\right) v_{j-1} \in \text{span}(v_1, v_2, \cdots, v_{j-1}),$$

$\Leftarrow$) If $v_j = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1}$

$0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{j-1} v_{j-1} + (-1) v_j \Rightarrow \{v_1, v_2, \cdots, v_j, v_{j+1}, \cdots, v_n\}$ is L.D.

$\blacksquare$

Proposition: Let V be an F-vector space, $\{v_1, v_2, \cdots, v_m\}$ a L.I. set. $\{w_1, w_2, \cdots, w_n\}$ spanning set, that is, $V = \text{span}(w_1, w_2, \cdots, w_n)$.

then $m \leq n$.

The cardinality of any L.I. set $\leq$ cardinality of any spanning set.

Example: $\mathbb{R}^2 = \{(x,y), x,y \in \mathbb{R}\} = \text{span}((1,0),(0,1))$ $n=2$

$\{(a,b),(c,d),(e,f)\}$ L.D $m = 3 \nleq 2$.

Proof: $V = \text{span}(w_1, w_2, \cdots, w_n)$, $B_0 = \{w_1, w_2, \cdots, w_n\}$, $\# B_0 = n$.

step 1: $v_1 \in V = \text{span}(w_1, w_2, \cdots, w_n) \Longleftrightarrow \{w_1, w_2, \cdots, w_n, v_1\}$ L.D. $\Longleftrightarrow$

$\exists j_1, j_1 > 1, (v_1 \neq 0)$, s.t. $w_{j_1} \in \text{span}(v_1, w_1, \cdots, w_{j_1 - 1})$.

$V = \text{span}(w_1, \cdots, w_n) = \text{span}(v_1, w_1, \cdots, w_n) = \text{span}(v_1, w_1, \cdots, w_{j_1-1}, w_{j_1+1}, \cdots, w_n)$

$B_1 = \{v_1, w_1, \cdots, w_{j_1-1}, w_{j_1+1}, w_n\}$, $\# B_1 = n$.

step 2: $v_2 \in V = \text{span}(v_1, w_1, \cdots, \hat{w}_{j_1}, \cdots, w_n) \Longleftrightarrow \{v_1, w_1, \cdots, \hat{w}_{j_1}, \cdots, w_n, v_2\}$ L.D.

$\Longleftrightarrow \exists j_2, j_2 > 2$ s.t. $w_{j_2} \in \text{span}(v_1, v_2, w_1, \cdots, w_{j_2 - 1})$.

$V = \text{span}(B_1) = \text{span}(v_1, v_2, w_1, \cdots, \hat{w}_{j_1}, \cdots, w_n) = \text{span}\underbrace{(v_1, v_2, w_1, \cdots, \hat{w}_{j_1}, \cdots, \hat{w}_{j_2}, \cdots, w_n)}_{B_2}$

step m: $v_m \in V = \text{span}(B_{m-1}) = \text{span}(v_1, v_2, \cdots, v_{m-1}, w_1, \cdots, \hat{w}_{j_1}, \cdots \hat{w}_{j_{m-1}}, \cdots, w_n)$.

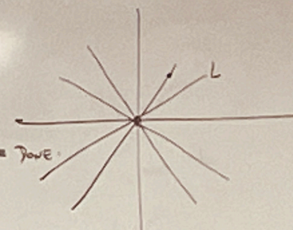
$\Longleftrightarrow \{\underbrace{v_1, v_2, \cdots, v_m}_{L.I.}, \underbrace{w_1, \cdots, \hat{w}_{j_1}, \cdots, \hat{w}_{j_{m-1}}, \cdots, w_n}_{\text{Cannot be empty}}\}$ L.D.

$m < \#(\{v_m\} \cup B_{m-1}) = 1 + n \Longleftrightarrow m \leq n$.

∎

APPLICATION (ALL) SUBSPACES OF $\mathbb{R}^2$ ARE $\{(0,0)\}$, $\mathbb{R}^2$ AND $L=\{\lambda(a,b),\ (a,b)\in\mathbb{R}^2\}$
$(a,b)\neq 0,\ \lambda\in\mathbb{R}$

FIRST CHECK THAT $\{(0,0)\}$, $\mathbb{R}^2$ AND L ARE SUBSPACES ✓



LET S BE A SUBSPACE

S SUBSPACE $\overset{(i)}{\Longrightarrow} (0,0)\in S \Longrightarrow \{(0,0)\}\subseteq S$. IF $\{(0,0)\}=S$ WE ARE DONE.

IF $\{(0,0)\}\subsetneq S \Longrightarrow \exists (a,b)\in\mathbb{R}^2,\ (a,b)\neq(0,0):\ (a,b)\in S$

S SUBSPACE, $(a,b)\in S \overset{(iii)}{\Longrightarrow} \lambda(a,b)\in S,\ \forall\lambda\in\mathbb{R} \Longrightarrow L=\{\lambda(a,b),\ \lambda\in\mathbb{R}\}\subseteq S$

IF $S=\{\lambda(a,b),\ \lambda\in\mathbb{R}\}$ WE ARE DONE.

IF $\{\lambda(a,b),\ (a,b)\neq(0,0),\ \lambda\in\mathbb{R}\}\subsetneq S \Longrightarrow \exists (c,d)\in S:\ (c,d)\neq\lambda(a,b),\ \forall\lambda\in\mathbb{R}$

$(c,d)\notin \text{SPAN}((a,b))$

IF $\{(0,0)\} \neq S \Longrightarrow \exists (a,b) \in \mathbb{R}, (a,b) \neq (0,0) : (a,b) \in S$

S SUBSPACE, $(a,b) \in S \overset{(1,1)}{\Longrightarrow} \lambda(a,b) \in S, \forall \lambda \in \mathbb{R} \Longrightarrow L = \{\lambda(a,b), \lambda \in \mathbb{R}\} \subseteq S$

IF $S = \{\lambda(a,b), \lambda \in \mathbb{R}\}$ WE ARE DONE.

IF $\{\lambda(a,b), (a,b) \neq (0,0), \lambda \in \mathbb{R}\} \subsetneq S \Longrightarrow \exists (c,d) \in S : (c,d) \neq \lambda(a,b), \forall \lambda \in \mathbb{R}$

$(c,d) \notin \text{SPAN}((a,b))$

$\Longrightarrow \{(a,b),(c,d)\}$ IS L I

$\mathbb{R}^2 = \{(x,y) = x(1,0) + y(0,1), x,y \in \mathbb{R}\} = \text{SPAN}((1,0),(0,1))$

$\Longrightarrow \{(a,b),(c,d),(x,y)\}$ L D $\quad (3 \nleq 2)$

$\Longrightarrow (x,y) \in \text{SPAN}((a,b),(c,d)) = \{\lambda_1(a,b) + \lambda_2(c,d), \lambda_1, \lambda_2 \in \mathbb{R}\} \subseteq S$

$\Longrightarrow \mathbb{R}^2 \subseteq S \Longrightarrow \mathbb{R}^2 \subseteq S \subseteq \mathbb{R}^2 \Longrightarrow S = \mathbb{R}^2.$

4.2 Lecture.

Theorem: Let V be a finite-dimension vector space. and let S be a subspace of V. Then S is also a finite dimension vector space.

Proof: V finite dimensional $\overset{Def}{\Longleftrightarrow}$ $\exists$ finite set $X \subseteq V$: $V = span(X)$, $X = \{w_1, \dots w_n\}$

If $S = \{0\} = span(\emptyset) \Rightarrow S$ is finite dimensional.

If not, $\exists v_1 \in S$, $v_1 \neq 0$ If $S = span(v_1) \Rightarrow S$ is finite dimensional

If not. $span(v_1) \subsetneq S \Rightarrow \exists v_2 \in S$, $v_2 \notin span(v_1) \Rightarrow \{v_1, v_2\}$ L.I.

$2 \leq n$

If $S = span(v_1, v_2) \Rightarrow S$ is F.D.

If not, $span(v_1, v_2) \subsetneq S$, $\exists v_3 \in S$. $v_3 \notin span(v_1, v_2)$. $\Rightarrow \{v_1, v_2, v_3\}$ L.I.

Since the cardinality of any L.I. set is $\leq n$, this algorith should stop at step k: $S = span(v_1, v_2, \dots, v_k) \Rightarrow S$ is F.D.

∎

Basis:

Definition: A basis for a vector space V is an ordered L.I. spanning set.

that is: B is a basis $\Longleftrightarrow$ B is L.I and $V = span(B)$.

Remark: $B = \{v_1, v_2, \dots, v_n\}$

$V = span(v_1, v_2, \dots, v_n) \Longleftrightarrow$ Any $v \in V$ can be written as a L.C. of $v_1, \dots, v_n$.

$v = \lambda_1 v_1 + \dots + \lambda_n v_n = \mu_1 v_1 + \dots + \mu_n v_n \Longleftrightarrow 0 = (\lambda_1 - \mu_1) v_1 + \dots + (\lambda_n - \mu_n) v_n$

B is L.I. $\Longleftrightarrow$ v can be written in a unique way.

Problem: How can we construct basis for vector spaces?

Theorem 1: Any spanning list of V contains a basis.

$$V = \text{span}(X) \Rightarrow \exists B \text{ basis}. \ B \subseteq X.$$

Theorem 2: Any L.I. set can be extended to a basis:

$$Y \subseteq V, \ Y \text{ is L.I.} \Rightarrow \exists B \text{ basis}, \ Y \subseteq B.$$

Proof: 1.): V finite dimensional, X finite $X = \{w_1, w_2, \dots, w_n\} = B_0$

step 1: If $w_1 = 0$, $B_1 = B_0 \setminus \{0\}$. $\Rightarrow \text{span}(B_1) = \text{span}(B_0)$

If not, $B_1 = B_0$

step 2: If $w_2 \in \text{span}(B_1)$, $B_2 = B_1 \setminus \{w_2\}$ $\Rightarrow \text{span}(B_2) = \text{span}(B_1)$.

If $w_2 \notin \text{span}(B_1)$, $B_2 = B_1$

⋮

step n: If $w_n \in \text{span}(B_{n-1}) \Rightarrow B_n = B_{n-1} \setminus \{w_n\}$

If $w_n \notin \text{span}(B_{n-1}) \Rightarrow B_n = B_{n-1}$.

$\Rightarrow B_n$ is a basis for V.

$V = \text{span}(X) = \text{span}(B_0) = \text{span}(B_1) = \text{span}(B_2) = \dots = \text{span}(B_{n-1}) = \text{span}(B_n)$

↑ INDUCTION

$\text{span}(B_n) = \text{span}(B_0) = \text{span}(X) \ \forall n$. $\Longrightarrow B_n$ IS A SPANNING LIST.

$n=1$: $\text{span}(B_0) \overset{\checkmark}{=} \text{span}(B_1)$ IH

TRUE FOR $n-1 \Longrightarrow \text{span}(B_0) = \text{span}(B_{n-1}) \overset{\checkmark}{=} \text{span}(B_n)$

B_n IS L.I.: ASSUME B_n IS L.D

$B_n = \{v_1, v_2, \dots, v_s\}$ L.D $\Longleftrightarrow \exists j: v_j \in \text{span}(v_1, \dots, v_{j-1})$.

BUT $v_j = w_k$, $w_k \in \text{span}(v_1, \dots, v_{j-1}) \subseteq \text{span}(B_{k-1})$. CONTRADICTION TO STEP k

4.7 Lecture.

Corollary: Any subspace S of a vector space V admits a complement, that is,
$\exists T \subseteq V$ a subspace such that $V = S \oplus T$.

Proof: S is a subspace of $V \Rightarrow S$ is a vector space $\Rightarrow \exists B_1$ a basis for S.
$\Rightarrow B_1$ is L.I. in S. $\Rightarrow B_1$ is L.I. in V. $\Rightarrow B_1 \subseteq B$, B a basis for V.
$B = B_1 \cup B_2$, $B_2 = B \setminus B_1$. Let $T = \text{span}(B_2)$.

- $S + T = \text{span}(B_1) + \text{span}(B_2) = \text{span}(B_1 \cup B_2) = \text{span}(B) = V$.
- $S \cap T = \{0\}$: Let $v \in S \cap T \Rightarrow v \in S \wedge v \in T$

set $B_1 = \{v_i, i \in I\}$ $B_2 = \{w_j, j \in J\}$.
$B = B_1 \cup B_2$ is a basis.
$B_1 \cap B_2 = \emptyset$.

$v = a_1 v_1 + \cdots + a_k w_k = b_1 w_1 + \cdots + b_s w_s$.
$0 = a_1 v_1 + \cdots + a_k b_k - b_1 w_1 - \cdots - b_s w_s$, $B_1 \cup B_2$ is L.I.
$\Rightarrow a_1 = \cdots = a_k = b_1 = \cdots = b_s = 0 \Rightarrow v = 0$.
$\Rightarrow V = S \oplus T$.

$\square$

Dimension:

Theorem: Let B_1 and B_2 be two basis for the F-vector space V.
Then $\# B_1 = \# B_2$.

Proof: Basis = L.I. + spanning set.
B_1 L.I., B_2 spanning set $\Rightarrow \# B_1 \leq \# B_2$
B_2 L.I., B_1 spanning set $\Rightarrow \# B_2 \leq \# B_1$.
then $B_1 = B_2$.

Def: $\dim_F V = \# B$, for B any basis for V. called the dimension of V.

Example: $\dim_{\mathbb{C}} \mathbb{C}^2$ $\quad \mathbb{C}^2 = \text{span}((0,1),(1,0))$, $(a+bi, c+di) = (a+bi)(1,0) + (c+di)(0,1)$.

$\{(0,1),(1,0)\}$ is L.I.

$\Rightarrow \dim_{\mathbb{C}} \mathbb{C}^2 = 2$

$\dim_{\mathbb{R}} \mathbb{C}^2$ $\quad \mathbb{C}^2 = \text{span}((1,0),(i,0),(0,1),(0,i))$ $(a+bi, c+di) = a(1,0)+b(i,0)+c(0,1)+d(0,i)$

$\{(1,0),(i,0),(0,1),(0,i)\}$ is L.I.

$\Rightarrow \dim_{\mathbb{R}} \mathbb{C}^2 = 4$.

Theorem: If S is a subspace of V then $\dim_F S \le \dim_F V$.

Proof: Let B_1 be a basis for S, B_2 a basis for V

B_1 is L.I. in $S \Rightarrow B_1$ is L.I. in V

B_2 is a spanning set for V.

$\Rightarrow \# B_1 \le \# B_2 \Rightarrow \dim_F S \le \dim_F V$.

$\blacksquare$

Theorem: Let V be an F-vector space of finite dimension. that is, $\dim_F V = n$.

let $B \subseteq V$, $\# B = n$. then the following are equivalent.

a) B is a basis for V.

b) B is L.I. in V

c) B is a spanning set for V.

Remark: Not true for not finite dimension

$\{x, x^2, x^3, \cdots\}$ is L.I. in $\mathbb{R}[x]$ but is not a spanning set.

Proof: a) ⇒ b) ✓ Basis ⟺ L.I. + spanning ⇒ L.I.

b) ⇒ c) B is L.I. in V, $\#B = n$

$\exists B_1$ basis : $B \subseteq B_1$, $\#B = n = \dim_F V = \#B_1 \Rightarrow B = B_1$

$\Rightarrow B$ spanning set.

c) ⇒ a) B is a spanning set $\Rightarrow \exists B_2$ basis : $B_2 \subseteq B$

$\#B_2 = \dim_F V = n = \#B \Rightarrow B_2 = B \Rightarrow B$ is a basis. ▨

Theorem: If $V = S_1 + S_2$, $\dim V < \infty$ then $\dim(S_1 + S_2) = \dim(S_1) + \dim(S_2) - \dim(S_1 \cap S_2)$

In particular, $\dim(S_1 \oplus S_2) = \dim S_1 + \dim S_2$.

Proof: Let $\{v_1, v_2, \cdots, v_r\}$ be a basis for $S_1 \cap S_2 \Rightarrow \dim S_1 \cap S_2 = r$

$\Rightarrow \{v_1, v_2, \cdots, v_r\}$ is L.I. in S_1 and S_2. we can extend this set to basis for S_1 and S_2

$B_1 = \{v_1, \cdots, v_r, u_1, \cdots, u_s\}$ basis for $S_1 \Rightarrow \dim S_1 = r + s$

$B_2 = \{v_1, \cdots, v_r, w_1, \cdots, w_k\}$ basis for $S_2 \Rightarrow \dim S_2 = r + k$.

W.T.P.: $\dim(S_1 + S_2) = r + s + k$.

Let's prove that $B = \{v_1, \cdots, v_r, u_1, \cdots, u_s, w_1, \cdots, w_k\}$ is a basis for $S_1 + S_2$.

$\text{span}(B) = \text{span}(B_1 \cup B_2) = \text{span}(B_1) + \text{span}(B_2) = S_1 + S_2 \Rightarrow B$ is a spanning set.

B is L.I. : $0 = a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_s u_s + c_1 w_1 + \cdots + c_k w_k$

$$\Leftrightarrow \underbrace{a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_s u_s}_{\in S_1} = \underbrace{-c_1 w_1 - \cdots - c_k w_k}_{\in S_2} = v \in S_1 \cap S_2.$$

$\Rightarrow S_1 \cap S_2 \ni -c_1 w_1 - \cdots - c_k w_k = \lambda_1 v_1 + \cdots + \lambda_r v_r$ since $\{v_1, \cdots, v_r\}$ is a basis for $S_1 \cap S_2$

$0 = \lambda_1 v_1 + \cdots + \lambda_r v_r + c_1 w_1 + \cdots + c_k w_k$. B_2 is L.I.

$\Rightarrow \lambda_1 = \cdots = \lambda_r = c_1 = \cdots = c_k = 0$

$\Rightarrow a_1 v_1 + \cdots + a_r v_r + b_1 u_1 + \cdots + b_s u_s = 0$ B_1 is L.I.

$\Rightarrow a_1 = \cdots = a_r = b_1 = \cdots = b_s = 0$. ▨

In particular, $\dim(S_1 \oplus S_2) = \dim S_1 + \dim S_2$ since $S_1 \cap S_2 = \{0\}$ and the basis for $\{0\}$ is $\emptyset$. Hence $\dim(S_1 \cap S_2) = 0$.

Example: $S_1 = \{(x, y, z, 0) : x, y, z \in \mathbb{R}\} \subseteq \mathbb{R}^4$

$S_2 = \{(x, 0, 0, w) : x, w \in \mathbb{R}\} \subseteq \mathbb{R}^4$

$S_1 \cap S_2 = \{(x, 0, 0, 0),\ x \in \mathbb{R}\} = \text{span}(\{1, 0, 0, 0\})$. $\quad \dim(S_1 \cap S_2) = 1$.

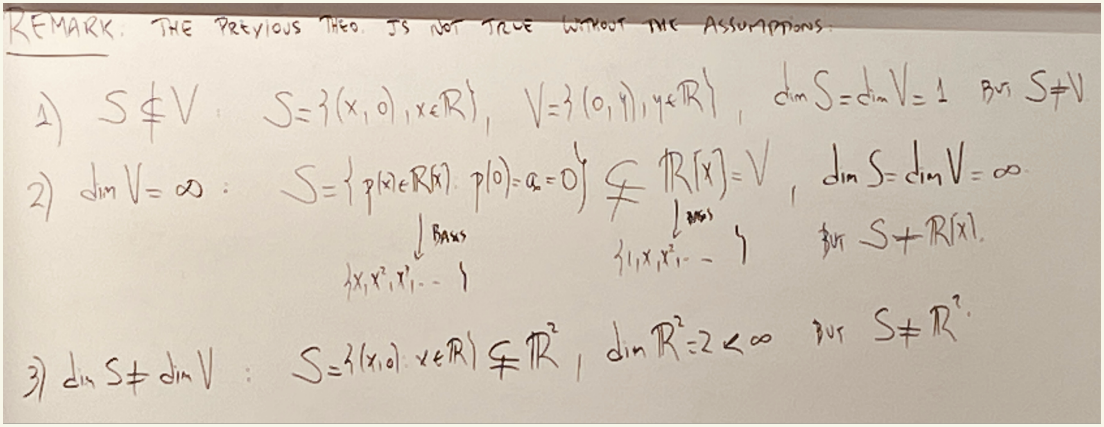
$S_1 = \text{span}(\{1,0,0,0\}, \{0,1,0,0\}, \{0,0,1,0\})$ $\quad \dim S_1 = 3$
L.I.

$S_2 = \text{span}(\{1,0,0,0\}, \{0,0,0,1\})$ $\quad \dim S_2 = 2$.
L.I.

$\dim(S_1 + S_2) = \dim S_1 + \dim S_2 - \dim S_1 \cap S_2 = 2 + 3 - 1 = 4$

$S_1 + S_2 \subseteq \mathbb{R}^4$, B a basis for $S_1 + S_2$, $\#B = 4$

Theorem: $S \subseteq V$, $\dim V < \infty$, then $\dim S = \dim V \Rightarrow S = V$. (In tutorial).



REMARK: THE PREVIOUS THEO. IS NOT TRUE WITHOUT THE ASSUMPTIONS:

1) $S \nsubseteq V$: $S = \{(x, 0), x \in \mathbb{R}\}$, $V = \{(0, y), y \in \mathbb{R}\}$, $\dim S = \dim V = 1$ BUT $S \neq V$

2) $\dim V = \infty$: $S = \{p(x) \in \mathbb{R}[x] : p(0) = a_0 = 0\} \subsetneq \mathbb{R}[x] = V$, $\dim S = \dim V = \infty$

BASIS of S: $\{x, x^2, x^3, \dots\}$; BASIS of $\mathbb{R}[x]$: $\{1, x, x^2, \dots\}$; BUT $S \neq \mathbb{R}[x]$.

3) $\dim S \neq \dim V$: $S = \{(x, 0) : x \in \mathbb{R}\} \subsetneq \mathbb{R}^2$, $\dim \mathbb{R}^2 = 2 < \infty$ BUT $S \neq \mathbb{R}^2$.

4.9 Lecture.

Matrices:

$M_{n\times m}(F)$ = set of matrices of order $n\times m$ with coefficients in F.

$A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \\ A_{n1} & \cdots & A_{nm} \end{pmatrix}$ = matrix in $M_{n\times m}(F)$ = set of $n.m$ elements in F arranged in n rows and m columns.

Operation:

Addition: $M_{n\times m}(F) \times M_{n\times m}(F) \xrightarrow{+} M_{n\times m}(F)$.

$(A, B) \longrightarrow A+B$: $(A+B)_{ij} = A_{ij} + B_{ij}$.

Properties: (S_1): Associtive (S_2): Commutative (S_3): $0 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix}$ (S_4). $-\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} = \begin{pmatrix} -A_{11} & \cdots & -A_{1m} \\ \vdots & & \vdots \\ -A_{n1} & \cdots & -A_{nm} \end{pmatrix}$.

Properties: 1) $\lambda(A+B) = \lambda A + \lambda B \in M_{n\times m}(F)$

$\forall i,j.$ $(\lambda\cdot(A+B))_{ij} = \lambda\cdot(A+B)_{ij} = \lambda(A_{ij}+B_{ij}) = \lambda A_{ij} + \lambda B_{ij} = (\lambda A)_{ij} + (\lambda B)_{ij} = (\lambda A+\lambda B)_{ij}$

2) $(\lambda+\mu)\cdot A = \lambda A + \mu A \quad \forall \lambda,\mu.\ \forall A.$

3) $(\lambda\cdot\mu)\cdot A = \lambda\cdot(\mu\cdot A)$.

4). $1\cdot A = A$.

Distributivity: $A\cdot(B+C) = A\cdot B + A\cdot C$.

Properties:

(P_1): Associativity: $(A.B).C = A.(B.C)$

(P_3): Identity: $\underset{n\times n}{I_n} \cdot \underset{n\times m}{A} = A = \underset{n\times m}{A} \cdot \underset{m\times m}{I_m}$.

Remark: (P_2) Commutativity is not true.

$$a \neq b \quad \left.\begin{matrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax & ay \\ bz & bt \end{pmatrix} \\ \begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ax & by \\ az & bt \end{pmatrix} \end{matrix}\right] \neq \text{ if } y \neq 0 \text{ or } z \neq 0.$$

(P_4): $A, \exists A^{-1}: A \cdot A^{-1} = Id$ Not true.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} x_1+y_1 & x_2+y_2 \\ 0 & 0 \end{pmatrix}.$$

Not true: $A \cdot B = 0 \not\Rightarrow A = 0$ or $B = 0$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Elementary operations on rows = Elementary matrices

Elementary operations:

1) $R_i \leftrightarrow R_j$ Interchange rows.

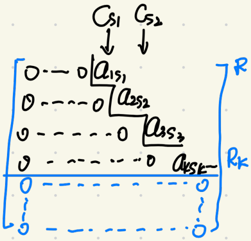
2) $R_i \rightarrow aR_i$ multiply row i by a

3) $R_i \rightarrow R_i + aR_j$ Replace R_i by $R_i + aR_j$.

4.21. Lecture

Def: $A, B \in M_{n\times m}(\mathbb{F})$ are row equivalent if $\exists\, E_1, E_2, \dots, E_s$ elementary matrices such that $E_s \cdots E_2 E_1 A = B$.

Def: A matrix $A \in M_{n\times m}(\mathbb{F})$ is called row echelon if:



1) The non zero rows appears first

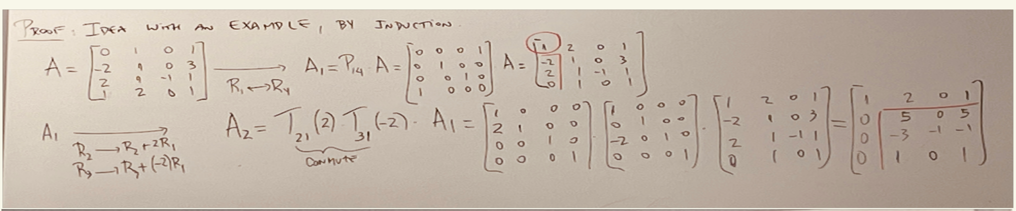
$R_1, R_2, \dots, R_k$ non zero, $R_{k+1} = R_{k+2} = \dots = R_n = 0$

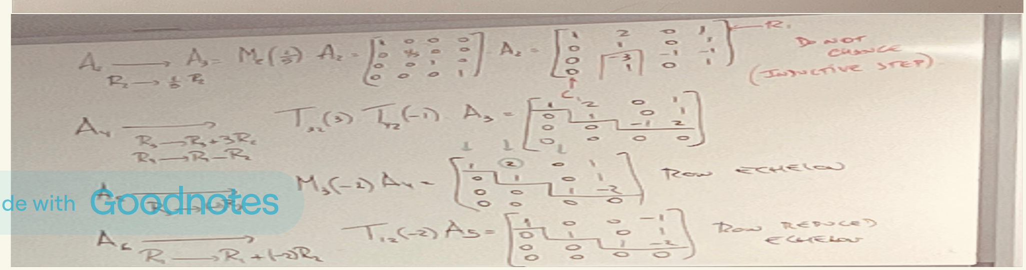
2) If the first non zero element in R_i appears in column S_i, then $S_1 < S_2 < \dots < S_k$

3) $a_{is_i} = 1, \ \forall_{1\le i\le k}$

Def: A matrix A is called row reduced echelon if it is row echelon and each column C_{s_i} has all its elements equit zero except $a_{is_i} = 1$.

Theorem: If $A \in M_{n\times m}(\mathbb{F})$ then there exists $E_1, E_2, \dots, E_k$ elementary matrices s.t. $E_\lambda \cdots E_2 E_1 A$ is row reduced echelon.





Definition: A system $Ax = b$ is called: S = Solutions for $\{(x_1, \cdots, x_m) \in F^m : Ax = b\}$.

1) Homogeneous: If $b = 0$
2) Inconsistent: If $S = \emptyset$
3) consistent: If $S \neq \emptyset$
4) consistent independent: $\#S = 1$
5) consistent dependent: $\#S > 1$

Remark: 1) Any homogeneous system is consistent since $(x_1, \cdots, x_m) = (0, \cdots, 0)$ is a solution

2) The set of solutions of any homogeneous system is a subspace of $\mathbb{F}^m$

3) If $b \neq 0$, S is not a subspace, since $(0, \cdots, 0) \notin S$.

Theorem: Let $S = \{(x_1, \cdots, x_m) \in F^m : Ax = b\}$, $S_0 = \{(x_1, \cdots, x_m) \in F^m : Ax = 0\}$ and let $z_0 \in S$

then $S = \{z_0 + w, w \in S_0\}$

Proof: $S \supseteq \{z_0 + w . w \in S_0\}$: $A(z_0 + w) = Az_0 + Aw = b + 0 = b$

$S \subseteq \{z_0 + w, w \in S_0\}$: $z_1 \in S$, $w = z_1 - z_0$: $Aw = A(z_1 - z_0) = b - b = 0$.

$\Rightarrow w \in S_0$ and $z_1 = z_0 + w$.

∎

Corollary: If S is a consistent dependent system then $\#F \leq \#S$?

In particular If $\#F = \infty$ then the possibilities for S are
- $\#S = 0$
- $\#S = 1$
- $\#S = \infty$.

Proof: If S is consistent dependent, take $z_1, z_2 \in S$, $z_1 \neq z_2$.

$\Rightarrow w = z_1 - z_2 \in S_0$, $w \neq 0$. Since S_0 is a subspace of F^m, then

$$\{\lambda w, \lambda \in F\} \subseteq S_0, \quad \#\{\lambda w, \lambda \in F\} = \#F \quad \text{since } F \longleftrightarrow \{\lambda w, \lambda \in F\}$$

$$\{z_0 + \lambda w, \lambda \in F\} \subseteq S, \quad \#\{z_0 + \lambda w, \lambda \in F\} = \#F. \qquad \lambda \longrightarrow \lambda w \text{ is bijective.}$$

$$\Rightarrow \#F \leq \#S.$$

∎

Theorem: Let $Ax = b$ be a system of linear equation. Consider $(A \mid b) \in M_{m\times(n+1)}(F)$ the matrix associated to the system if $(A' \mid b')$ is obtained from $(A \mid b)$ by applying row operations, then

$$S = \{(x_1, \ldots, x_m) \in F^m : Ax = b\} = S' = \{(x_1, \ldots, x_m) \in F^m : A'x = b'\}$$

Proof: $(A \mid b) \xrightarrow[E_1]{} \xrightarrow[E_2]{} \cdots \xrightarrow[E_3]{} (A' \mid b') \Longleftrightarrow E_3 \cdots E_1 (A \mid b) = (A' \mid b')$

$$\Longleftrightarrow \begin{cases} E_3 \cdots E_1 A = A' \\ E_3 \cdots E_1 b = b' \end{cases}$$

$S \subseteq S'$ if $Ax = b \Rightarrow A'x = E_3 \cdots E_1 A \cdot x = E_3 \cdots E_1 b = b' \Rightarrow A'x = b'$

$S \supseteq S'$ if $\begin{cases} E_3 \cdots E_1 A = A' \\ E_3 \cdots E_1 b = b' \end{cases} \Rightarrow \begin{cases} E_1^{-1} \cdots E_3^{-1} A' = A \\ E_1^{-1} \cdots E_3^{-1} b' = b \end{cases}$, then $A'x = b' \Rightarrow Ax = E_1^{-1} \cdots E_3^{-1} A'x = E_1^{-1} \cdots E_3^{-1} b' = b$.

∎

Linear maps ?

F - vector spaces

V, W are F - vector spaces, we want to compare them.

Def: A linear map from V to W is a function $T: V \to W$ such that

$$T(v_1 + v_2) = T(v_1) + T(v_2) \quad \forall v_1, v_2 \in V$$

$$T(\lambda \cdot v) = \lambda \cdot T(v) \quad \forall \lambda \in F,\ v \in V.$$

Notation: $\mathcal{L}_F(V, W) = \mathrm{Hom}_F(V, W)$ = set of all linear maps from V to W. for V, W two F-vector spaces.

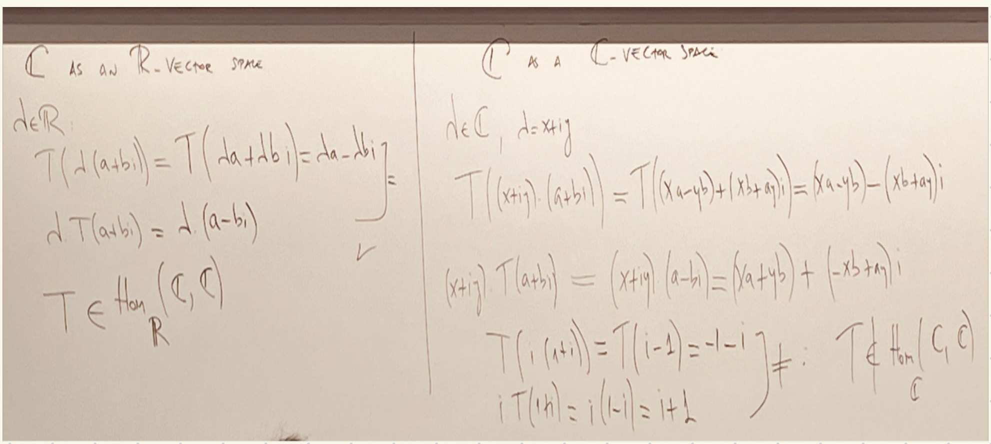
Example: 1) $T: V \to W$, $T(v) = 0_W$ is a linear map.

2) $Id: V \to V$, $Id(v) = v$.

3) $T: \mathbb{C} \to \mathbb{C}$, $T(a+bi) = a - bi$

Check $T \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$

But $T \notin \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$.



4.23 Lecture.

Theorem: Let $\{v_1, v_2, \cdots, v_n\}$ be a basis for V, let $\{w_1, w_2, \cdots, w_n\}$ in W, V, W $\mathbb{F}$ - vector space. Then $\exists!\ T: V \to W$ $\mathbb{F}$ - linear map such that $T(v_i) = w_i$

Proof: $\exists$ $T(v) = T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1w_1 + a_2w_2 \cdots + a_nw_n$

well defined since $v = a_1v_1 + \cdots + a_nv_n$ in a unique way.

T is a linear map:

$$T(v+u) = T(a_1v_1 + \cdots + a_nv_n + b_1v_1 + \cdots + b_nv_n) = T((a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n) = (a_1+b_1)w_1 + \cdots + (a_n+b_n)w_n$$

$$= \underbrace{a_1w_1 + \cdots + a_nw_n}_{v} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{u}$$

$$= T(v) + T(u).$$

$$T(\lambda v) = T(\lambda(a_1v_1 + \cdots + a_nv_n)) = T(\lambda a_1v_1 + \cdots + \lambda a_nv_n) = \lambda a_1w_1 + \cdots + \lambda a_nw_n$$

$$= \lambda(a_1w_1 + \cdots + a_nw_n)$$

$$= \lambda \cdot T(v).$$

$$T(v_i) = T(0\cdot v_1 + 0\cdot v_2 + \cdots + 1\cdot v_i + 0\cdot v_{i+1} + \cdots + 0\cdot v_n) = 0\cdot w_1 + \cdots + 1\cdot w_i + \cdots + 0\cdot w_n = w_i$$

$!$: Let $T \in \mathrm{Hom}_{\mathbb{F}}(V, W)$ such that $T(s_i) = w_i$ $\forall i = 1, \cdots, n$

$$T(v) = T(a_1v_1 + \cdots + a_nv_n) = T(a_1v_1) + \cdots + T(a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n)$$

$$= a_1w_1 + \cdots + a_nw_n$$

Example: There is no linear map $T: \mathbb{R}^2 \to \mathbb{R}^3$. $T(1,0) = (1,2,3)$ $T(0,1) = (3,2,1)$ $T(1,1) = (0,1,0)$

If $T \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^3)$ $T(x,y) = T(x,0) + T(0,y) = xT(1,0) + yT(0,1)$

$$= x(1,2,3) + y(3,2,1) = (x+3y, 2x+2y, 3x+y)$$

$x, y = 1 \Rightarrow T(1,1) = (4,4,4) \neq (0,1,0)$.

I want to define algebraic structures on the set $Hom_F(V, W)$.

Addition:

$$Hom_F(V,W) \times Hom_F(V,W) \xrightarrow{+} Hom_F(V,W)$$

$$(T_1, T_2) \longrightarrow T_1+T_2 : V \longrightarrow W$$

$$(T_1+T_2)(v) = T_1(v) + T_2(v) \quad \text{Function}$$

$T_1 + T_2 \in Hom_F(V, W)$:

$$(T_1+T_2)(v_1+v_2) \stackrel{Def}{=} T_1(v_1+v_2) + T_2(v_1+v_2) \stackrel{L.M}{=} T_1(v_1) + T_1(v_2) + T_2(v_1) + T_2(v_2)$$

$$\underset{Comm.}{=} T_1(v_1) + T_2(v_1) + T_1(v_2) + T_2(v_2) \underset{Def}{=} (T_1+T_2)(v_1) + (T_1+T_2)(v_2).$$

$$(T_1+T_2)(\lambda v) = T_1(\lambda v) + T_2(\lambda v) = \lambda T_1(v) + \lambda T_2(v) = \lambda(T_1(v)+T_2(v)) = \lambda(T_1+T_2)(v)$$

properties: Associativity: $(T_1+T_2)+T_3 = T_1+(T_2+T_3)$

Comm. $T_1 + T_2 = T_2 + T_1$

identity $\hat{0} + T = T = T + \hat{0}$, $\hat{0} : V \to W$, $\hat{0}(v) = 0_W$

Inverse: $-T$ as $(-T)(v) = -T(v)$.

product by scalars

$$F \times Hom_F(V,W) \xrightarrow{\cdot} Hom_F(V,W)$$

$$(\lambda, T) \longrightarrow \lambda \cdot T : V \to W$$

$$(\lambda \cdot T)(v) = \lambda \cdot T(v) \quad \text{Function}$$

λT is a linear map:

$$(\lambda T)(v_1+v_2) = \lambda \cdot T(v_1+v_2) = \lambda(T(v_1)+T(v_2)) = \lambda T(v_1) + \lambda T(v_2) = (\lambda T)(v_1) + (\lambda T)(v_2)$$

$$(\lambda T)(\mu v) = \lambda \cdot T(\mu v) = \lambda(\mu T(v)) = \lambda\mu \cdot T(v) = \mu(\lambda T(v)) = \mu \cdot (\lambda \cdot T)(v).$$

Composition of linear maps:

$$\mathrm{Hom}_F(V,W)\times \mathrm{Hom}_F(W,U) \longrightarrow \mathrm{Hom}_F(V,U)$$
$$(f,g) \longrightarrow g\circ f : V\to U. \quad \text{Function}$$

$g\circ f$ is a linear map

$$(g\circ f)(v_1+v_2)=g(f(v_1+v_2)) = g(f(v_1)+f(v_2)) = g(f(v_1))+g(f(v_2)).$$

$$(g\circ f)(\lambda v) = g(f(\lambda v)) = g(\lambda f(v)) = \lambda\cdot g(f(v)).$$

Def: Let $T\in \mathrm{Hom}_F(V,W)$

1) $\mathrm{Ker}\,T = \mathrm{Null}\,T = \text{Kernel of } T = \text{Nullspace of } T = \{v\in V : T(v)=0\} \subseteq V.$

2) $\mathrm{Im}\,T = \mathrm{Range}\,T = \text{Image of } T = \text{Range of } T = \{T(v),\ v\in V\} \subseteq W.$

Remark: If $T: F^n \to F^m$ is the linear map associated to a system of linear equations; $T(x_1,\cdots,x_n) = (a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \cdots, a_{m1}x_1 + \cdots a_{mn}x_n)$ then

$$\mathrm{Ker}\,T = \left\{(x_1,\cdots,x_n) : \begin{cases} a_{11}x_1+\cdots+a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1+\cdots+a_{mn}x_n=0\end{cases}\right\} = \text{solution of the Homogeneous system } Ax=0$$

$$(b_1,\cdots,b_n)\in \mathrm{Im}\,T \iff (b_1,\cdots,b_n) = T(x_1,\cdots,x_n) = (a_{11}x_1+\cdots+a_{1n}x_n,\cdots,a_{m1}x_1+\cdots+a_{mn}x_n) \iff$$

$Ax=b$ is consistent.

Made with Goodnotes

4.28 Lecture

Compute $\ker T$ and $\operatorname{Im} T$

$\ker T = \{(x,y) : T(x,y) = (0,0,0)\} = \{(x,y) : (2x-3y, x+y, 3x) = (0,0,0)\}$.

$= \left\{(x,y) : \begin{cases} 2x-3y=0 \\ x+y=0 \\ 3x=0 \end{cases}\right\}$ = set of solutions of $A\bar{x} = 0$.

$= \{(0,0)\}$ subspace of $\mathbb{R}^2$, $\dim \ker T = 0$

$\operatorname{Im} T = \{T(x,y) : (x,y) \in \mathbb{R}^2\} = \{(2x-3y, x+y, 3x),\ x,y \in \mathbb{R}\}$. subspace of $\mathbb{R}^3$

$(2x-3y, x+y, 3x) = x(2,1,3) + y(-3,1,0)$ $\quad \{(2,1,3), (-3,1,0)\}$ is L.I. $\dim \operatorname{Im} T = 2$

$2 = \dim \mathbb{R}^2 = \dim \ker T + \dim \operatorname{Im} T$.

Proposition: Let $T \in \operatorname{Hom}_F(V, W)$.

1) $\ker T$ is a subspace of V.
2) $\operatorname{Im} T$ is a subspace of W.

Proof: 1) · $(0,0) \in \ker T$ since $T(0,0) = (2\cdot 0 - 3\cdot 0,\ 0+0,\ 3\cdot 0) = (0,0,0)$

· $(x,y), (x',y') \in \ker T \overset{?}{\Longrightarrow} (x,y) + (x',y') \in \ker T$

$T((x,y)+(x',y')) = T(x,y) + T(x',y') = (0,0,0) + (0,0,0) = (0,0,0)$.

· $\lambda \in \mathbb{R}$, $(x,y) \in \ker T \overset{?}{\Longrightarrow} \lambda(x,y) \in \ker T$

$T(\lambda(x,y)) = \lambda T(x,y) = \lambda(0,0,0) = (0,0,0)$.

$\Rightarrow \ker T$ is a vector space.

2) $\operatorname{Im} T = \{T(x,y) : (x,y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^3$

- $(0,0,0) = T(0,0) \in \operatorname{Im} T$
- $(w_1, w_2, w_3), (u_1, u_2, u_3) \in \operatorname{Im} T \overset{?}{\Longrightarrow} (w_1, w_2, w_3) + (u_1, u_2, u_3) \in \operatorname{Im} T$.

  $(w_1, w_2, w_3) + (u_1, u_2, u_3) = T(x,y) + T(x',y') = T(x+x', y+y') \in \operatorname{Im} T$
- $\lambda \in \mathbb{R}, (w_1, w_2, w_3) \in \operatorname{Im} T \overset{?}{\Longrightarrow} \lambda(w_1, w_2, w_3) \in \operatorname{Im} T$

  $\lambda(w_1, w_2, w_3) = \lambda \cdot T(x,y) = T(\lambda(x,y)) \in \operatorname{Im} T$

$\boxtimes$

Lemma: If $\{v_1, v_2, \dots, v_n\}$ is a basis of V, $T \in \operatorname{Hom}_F(V, W)$ then

$$\operatorname{Im} T = \operatorname{span}\{T(v_1), T(v_2), \dots, T(v_n)\}.$$

Proof: We have to prove that.

$$\{T(v), v \in V\} = \operatorname{Im} T \overset{?}{=} \operatorname{span}(T(v_1), \dots, T(v_n)) = \text{set of L.C. of } T(v_1), \dots, T(v_n).$$

$v \in V$, $\{v_1, \dots, v_n\}$ is a basis for V

$$v = a_1 v_1 + \dots + a_n v_n$$

$$T(v) = T(a_1 v_1 + \dots + a_n v_n) = a_1 T(v_1) + \dots + a_n T(v_n).$$

($\supseteq$, $\subseteq$)

Theorem: Let $T \in \operatorname{Hom}_F(V, W)$, $\dim_F V < \infty$. Then $\dim_F V = \dim_F \ker T + \dim_F \operatorname{Im} T$.

Proof: We construct a basis for $\ker T$: $\{v_1, v_2, \dots, v_r\}$ ($\{v_1, v_2, \dots, v_r\}$ is L.I. in V)

We extend it to a basis of V : $\{v_1, v_2, \dots, v_r, v_{r+1}, \dots, v_n\}$, $n = \dim V$, $r = \dim \ker T$.

$\operatorname{Im} T$ is spanned by the set $\{T(v_1) = 0_W, T(v_2) = 0_W, \dots, T(v_r) = 0_W, T(v_{r+1}), \dots, T(v_n)\}$

$\operatorname{Im} T = \operatorname{span}(T(v_1), \dots, T(v_r), T(v_{r+1}), \dots, T(v_n)) = \operatorname{span}(T(v_{r+1}), \dots, T(v_n))$.

$T(v) = T(a_1v_1 + a_2v_2 + \cdots + a_{r+1}v_{r+1} + \cdots + a_nv_n) = \underbrace{a_1T(v_1) + \cdots + a_rT(v_r)}_{0} + a_{r+1}T(v_{r+1}) + \cdots + a_nT(v_n)$.

$\Rightarrow \{T(v_{r+1}), \cdots, T(v_n)\}$ is a spanning set for $\operatorname{Im} T$.

We have to prove that $\{T(v_{r+1}), \cdots, T(v_n)\}$ is L.I. in W

Let $0_W = b_{r+1}T(v_{r+1}) + \cdots + b_nT(v_n) \overset{?}{\Longrightarrow} b_{r+1} = \cdots = b_n = 0$

$0_W = T(b_{r+1}v_{r+1} + \cdots + b_nv_n) \Longrightarrow b_{r+1}v_{r+1} + \cdots + b_nv_n \in \ker T$

And $\{v_1, \cdots, v_r\}$ is a basis for $\ker T$. Then

$$b_{r+1}v_{r+1} + \cdots + b_nv_n = c_1v_1 + c_2v_2 + \cdots + c_rv_r$$

$\Rightarrow (-c_1)v_1 + \cdots + (-c_r)v_r + b_{r+1}v_{r+1} + \cdots + b_nv_n = 0$ Since $\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ is L.I.

$\Rightarrow c_1 = \cdots = c_r = 0$, $b_{r+1} = \cdots = b_n = 0 \Rightarrow \dim \operatorname{Im} T = n - r$.

∎.

Def: Let $T \in \operatorname{Hom}_F(V, W)$.

1) T is a monomorphism $\Leftrightarrow$ T is an injective function

2) T is an epimorphism $\Leftrightarrow$ T is a surjective function

3) T is an isomorphism $\Leftrightarrow$ T is bijective function.

Proposition: Let $T \in \operatorname{Hom}_F(V, W)$

1) T is a monomorphism $\Leftrightarrow \ker T = \{0_V\}$

2) T is an epimorphism $\Leftrightarrow \operatorname{Im} T = W$

3) T is an isomorphism $\Leftrightarrow \ker T = \{0\}$ and $\operatorname{Im} T = W$.

$\Leftrightarrow \exists T^{-1}: W \to V$, $T^{-1} \in \operatorname{Hom}_F(V, W)$: $T \circ T^{-1} = id_W$, $T^{-1} \circ T = id_V$

Proof: 1) $\Leftarrow$) Assume $\ker T = \{0_V\}$, prove that T is injective.

$T(v_1) = T(v_2) \Rightarrow 0 = T(v_1) - T(v_2) = T(v_1 - v_2) \Rightarrow v_1 - v_2 \in \ker T = \{0_V\} \Rightarrow v_1 = v_2$
$= 0_V$.

$\Rightarrow$) We know that $\{0_V\} \subseteq \ker T$ since $T(0_V) = 0_W$

Let $v \in \ker T \Rightarrow T(v) = 0_W = T(0_V) \underset{T \text{ injective}}{\Longrightarrow} v = 0_V$.

2) By definition.

T surjective $\Leftrightarrow \operatorname{Im} T = \{T(v),\ v \in V\} = W$

3) By (1) and (2), T bijective $\Leftrightarrow T$ injective and surjective $\Leftrightarrow \ker T = \{0_V\}$ and $\operatorname{Im} T = W$

We will prove : T isomorphism $\begin{pmatrix} T \text{ linear map} \\ T \text{ bijective function} \end{pmatrix} \Longleftrightarrow T$ has an inverse $T^{-1} \in \operatorname{Hom}_F(V, W)$

T bijective function $\Longleftrightarrow T^{-1}$ has an inverse function

$\Leftarrow$) If T has inverse $\Rightarrow T$ is bijective

We have to prove that T^{-1} is a linear map

$$T^{-1}(w+w') = T^{-1}(T(v)+T(v')) = T^{-1}(T(v+v')) = v+v' = T^{-1}(w) + T^{-1}(w').$$

$$T^{-1}(\lambda w) = T^{-1}(\lambda \cdot T(v)) = T^{-1}(T(\lambda v)) = \lambda v = \lambda \cdot T^{-1}(w).$$

Theorem : Let $T \in \operatorname{Hom}_F(V, W)$, $\dim_F V = n$, $\dim_F W = m$. $T: V \to W$

a) If T is an epimorphism $\Rightarrow n \geq m$

b) If T is a monomorphism $\Rightarrow n \leq m$

c) If T is an isomorphism $\Rightarrow n = m$

Proof: ?

a) T epimorphism $\Leftrightarrow \operatorname{Im} T = W \Rightarrow n = \dim \ker T + \dim \operatorname{Im} T = \dim \ker T + \dim W$

$= \dim \ker T + m \geq m$

b) T monomorphism $\iff \ker T = \{0\} \implies n = \dim V = \dim \ker T + \dim \operatorname{Im} T$

$$= \dim \operatorname{Im} T \leq \dim W = m$$

$$\implies n \leq m$$

c) T isomorphism $\iff T$ mono and epi $\iff n \leq m \wedge n \geq m \iff n = m$. ∎

Remark :

$n \geq m \not\Rightarrow T$ epi

$n \leq m \not\Rightarrow T$ mono

$n = m \not\Rightarrow T$ iso

4.30. Lecture.

Corollary: If $T \in Hom_F(V, W)$, $\dim V = \dim W = n$, then following are equivalent

a) T is an isomorphism

b) T is a monomorphism

c) T is an epimorphism

Proof: a ⇒ b) T iso ⇒ T mono and epi ⇒ T mono

b ⇒ c) Assume T is a monomorphism ⇒ $\ker T = \{0\}$ ⇒ $\dim \ker T = 0$

By theo, $n = \dim V = \dim \ker T + \dim \operatorname{Im} T = \dim \operatorname{Im} T$.

$\operatorname{Im} T \subseteq W$, $\dim \operatorname{Im} T = n = \dim W \Rightarrow \operatorname{Im} T = W \Rightarrow T$ epi.

c ⇒ a) Assume T is epi. then $\operatorname{Im} T = W \Rightarrow \dim \operatorname{Im} T = \dim W = n$

By theo, $n = \dim V = \dim \ker T + \dim \operatorname{Im} T = \dim \ker T + n \Rightarrow \dim \ker T = 0 \Rightarrow \ker T = \{0\}$

⇒ T mono ⇒ T epi and mono ⇒ T iso.

∎

Remark: The previous theorem is not true. If $\dim V = \dim W = \infty$

1) $T: F[x] \longrightarrow F[x]$

$p(x) \longrightarrow p'(x)$ Linear map ✓

T not mono: $T(\lambda) = 0 \quad \forall \lambda \in F$

Made with Goodnotes

Example : $T: \mathbb{R}^3 \to \mathbb{R}^3, T(x,y,z) = (x+y+z, x-z, x-y)$.

a). T is a linear map

b). T is epi.

$\operatorname{Im} T = \operatorname{span}(T(1,0,0), T(0,1,0), T(0,0,1)) = \operatorname{span}((1,1,1), (1,0,-1), (1,-1,0)) \subseteq \mathbb{R}^3$.

$a(1,1,1) + b(1,0,-1) + c(1,-1,0) = (0,0,0) \iff \begin{cases} a+b+c=0 \\ a-c=0 \\ a-b=0 \end{cases} \iff \begin{cases} a=b=c \\ 3a=0 \end{cases} \iff (a,b,c) = (0,0,0)$

$\dim \operatorname{Im} T = 3$ since $\{(1,1,1), (1,0,-1), (1,-1,0)\}$ is a basis for $\operatorname{Im} T$.

$\operatorname{Im} T \subseteq \mathbb{R}^3$, $\dim \operatorname{Im} T = 3 = \dim \mathbb{R}^3 \Rightarrow \operatorname{Im} T = \mathbb{R}^3 \Rightarrow T$ epi.

How can we detect : monomorphism using L.I. set.
epimorphism using spanning set.
isomorphism using basis.

Theorem 1 : $T: V \to W$ is a monomorphism $\iff$ T transforms L.I. set. into L.I. set.

For any $X \subseteq V$. X L.I. $\iff T(X)$ is L.I. in W

Theorem 2 : $T: V \to W$ is an epimorphism $\iff$ It transforms spanning set of V into spanning set of W : $\forall Y \subseteq V$: $V = \operatorname{span}(Y)$ then $\operatorname{span}(T(V)) = W$.

Theorem 3 : $T \in \operatorname{Hom}_F(V, W)$ then F.A.E.

1) T is an isomorphism

2) T transforms any basis B of V into a basis $T(B)$ of W.

3) $\exists B$ basis for V s.t. $T(B)$ is a basis of W.

Proof: Theorem 1:

$\Rightarrow$) Assume T is a monomorphism. We will prove that $\{v_1, v_2, \dots, v_n\}$ is L.I. $\Leftrightarrow$ $\{T(v_1), T(v_2), \dots, T(v_n)\}$ is L.I.

$$\underbrace{a_1T(v_1) + a_2T(v_2) + \dots + a_nT(v_n) = 0}_{(*)} \Leftrightarrow T(a_1v_1 + a_2v_2 + \dots + a_nv_n) = 0 \underset{T \text{ mono}}{\Leftrightarrow} \underbrace{a_1v_1 + \dots + a_nv_n = 0}_{(**)}.$$

$\Rightarrow$) If $\{v_1, \dots, v_n\}$ is L.I., then $(*) \Leftrightarrow (**) \Rightarrow a_1 = \dots = a_n = 0$.

$\Leftarrow$) If $\{T(v_1), \dots, T(v_n)\}$ is L.I., then $(**) \Leftrightarrow (*) \Rightarrow a_1 = \dots = a_n = 0$.

$\Leftarrow$) Assume that X L.I. in V $\Leftrightarrow$ $T(X)$ L.I. in W. We will prove that T is monomorphism. Assume T is not a monomorphism, $\exists v \neq 0$, $T(v) = 0$, then $X = \{v\}$ is L.I. set. $T(X) = \{T(v) = 0\} = \{0\}$ Not L.I. Contradiction!

Theorem 2:

$\Rightarrow$) Assume T epi. We will prove: $V = \text{span}(Y) \Rightarrow W = \text{span}(T(Y))$.

Let $V = \text{span}(v_1, v_2, \dots, v_k)$ $\quad v = a_1v_1 + \dots + a_kv_k \Rightarrow T(v) = a_1T(v_1) + \dots + a_kT(v_k)$, $\forall v \in V$

$$\Rightarrow \text{span}(T(v_1), T(v_2), \dots, T(v_k)) = \text{Im}\, T \underset{T \text{ epi}}{=} W$$

$\Leftarrow$) Assume $V = \text{span}(Y) \Rightarrow W = \text{span}(T(Y))$. We will prove that T is epimorphism

By contradiction, T not epi then $\text{Im}\, T \subsetneq W$

Take $Y = V \Rightarrow V = \text{span}(V)$ but $\text{span}(T(V)) = \text{Im}\, T = W$

Theorem 3:

From theorem 1 and theorem 2, we know that T is isomorphism $\Leftrightarrow$ For any basis B of V, $T(B)$ is L.I. and $T(B)$ is spanning set

that is, $(1) \Leftrightarrow (2)$

$(2) \Rightarrow (3)$ is clear: True for any basis $\Rightarrow$ true for one basis

(3) $\Rightarrow$ (1) Assume $\{v_1, \cdots, v_n\}$ is a basis of V and $\{T(v_1), \cdots, T(v_n)\}$ is a basis for W. T epi: $W = \operatorname{Im} T$, $w \in W$, $w = a_1 T(v_1) + \cdots + a_n T(v_n)$

$$= T(a_1 v_1 + \cdots + a_n v_n) \in \operatorname{Im} T \Rightarrow W \subseteq \operatorname{Im} T$$

We know $\operatorname{Im} T \subseteq W$.

T mono: $v \in \ker T$, $v = a_1 v_1 + \cdots + a_n v_n \Rightarrow 0 = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n)$

$$\Rightarrow a_1 = a_2 = \cdots = a_n = 0 \Rightarrow v = 0 \Rightarrow T \text{ mono}.$$

$\blacksquare$

Prop: $\dim V = n$, $\dim W = m$. If $n \le m$, then $\exists\, T: V \to W$ such is a monomorphism.

Proof: Take $\{v_1, \cdots, v_n\}$ a basis for V. take $\{w_1, \cdots, w_n\}$ a L.I. set in W

Define $T: V \to W$ as the linear map $T(v_i) = w_i \ \forall i$.

T mono: $T(v) = 0 \Rightarrow v = a_1 v_1 + \cdots + a_n v_n \Rightarrow 0 = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n)$

$$= a_1 w_1 + \cdots + a_n w_n$$

$$\Rightarrow a_1 = \cdots = a_n = 0 \Rightarrow v = 0 \Rightarrow T \text{ mono}$$

$\blacksquare$

5.7 Lecture.

Def: $A \in M_{m\times n}(F)$

$\text{row rank}(A) = \dim_F \text{span}(R_1, R_2, \dots, R_m)$

$\text{column rank}(A) = \dim_F \text{span}(C_1, C_2, \dots, C_n)$

Example: $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 4 \end{pmatrix}$

$\text{row rank}(A) = \dim_F ((1,2,-1,0),(0,1,2,4)) = 2$

$\text{column rank}(A) = \dim_F (\underbrace{(1,0),(2,1)}_{L.I.},(-1,2),(0,4)) = 2$

Theorem: row rank (A) = column rank (A)

Proof: Let A' be a row reduced echelon form of A.

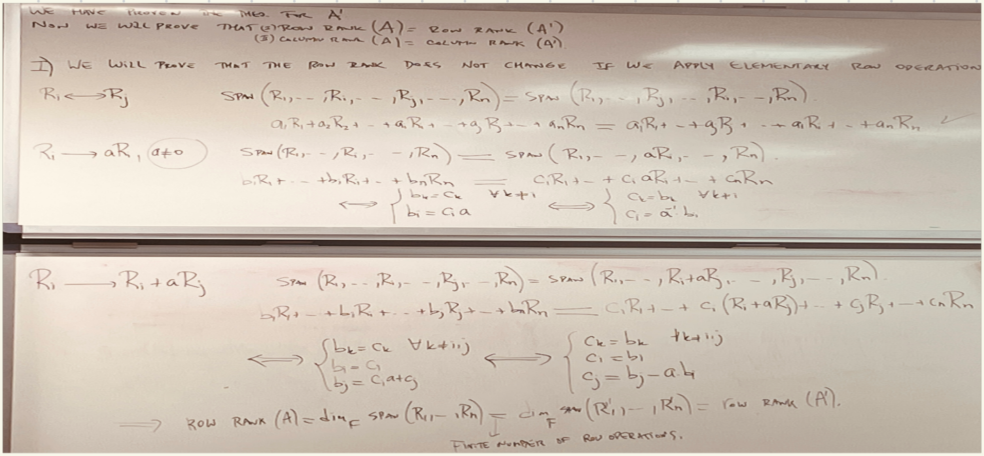
$$A \xrightarrow[\text{operations}]{\text{Finite number of row}} A' = \begin{pmatrix} 0 \cdots a_{1s_1} & 0 & 0 & 0 \\ 0 \cdots 0 \cdot a_{2s_2} & 0 & & \\ 0 \cdots & & & \\ \cdots & a_{is_i} & & \\ & & & \end{pmatrix}$$

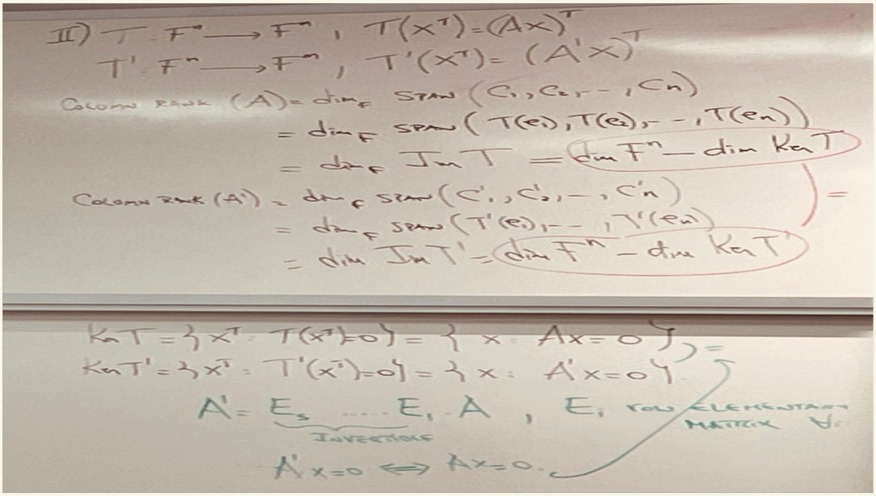
$a_{is_i} = 1$

$\text{row rank}(A') = \dim_F ((0 \cdots 0, a_{1s_1}, \cdots), (0, \cdots, 0, a_{2s_2}, \cdots), \cdots, (0, \cdots, 0, a_{ks_k}, \cdots)) = k$

$\text{column rank}(A') = \dim_F (\underset{\overset{\shortparallel}{e_1}}{C_{s_1}}, \underbrace{C_{s_1+1}, \cdots}_{L.D.}, \underset{\overset{\shortparallel}{e_2}}{C_{s_2}}, C_{s_2+1}, \cdots C_{s_k}, \underbrace{C_{s_k+1}, \cdots, C_n}_{L.D.}) = k.$

∎





So : $\text{rank}(A) = \text{row rank}(A) = \text{column rank}(A)$.

Theorem : let $A \in M_{m \times n}(F)$, $Ax = b$ a system of m linear equations with n unknowns

1) The system is inconsistent $\iff \text{rank}(A) < \text{rank}(A \,\vdots\, b)$

2) " " consistent independent $\iff \text{rank}(A) = \text{rank}(A \,\vdots\, b) = n$

3) " " consistent dependent $\iff \text{rank}(A) = \text{rank}(A \,\vdots\, b) < n$.

EXAMPLE: $\begin{cases} x+2y+z=3 \\ 2x-y+z=1 \\ 3x+y+2z=a \end{cases}$

$A=\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $\text{Rank}(A)=\dim \text{span}(\underbrace{(1,2,1),(2,-1,1)}_{LI},(3,1,2)=(1,2,1)+(2,-1,1))=2$

$A\vdots b=\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & 2 & a \end{array}\right]$, $\text{Rank}(A\vdots b)=\dim \text{span}(\underbrace{(1,2,1,3),(2,-1,1,1)}_{LI},(3,1,2,a))=$

$$=\begin{cases} 2 & \text{IF } (3,1,2,a)=\lambda(1,2,1,3)+\mu(2,-1,1,1) \iff \begin{cases} 3=\lambda+2\mu \\ 1=2\lambda-\mu \\ 2=\lambda+\mu \\ a=3\lambda+\mu \end{cases} \Rightarrow \mu=1,\ \lambda=1,\ a=4 \\ 3 & \text{IF NOT } (a\neq 4) \end{cases}$$

INCONSISTENT $\iff a\neq 4$

CONSISTENT DEPENDENT $\iff a=4$

Proof: $Ax=b \rightsquigarrow T: F^n \to F^m$, $T(x^T)=(Ax)^T$

$$\text{rank } A = \dim \text{span}(C_1, C_2, \dots, C_n) = \dim \text{span}(T(e_1), T(e_2), \dots, T(e_n)) = \dim \text{Im}\, T$$

$$\leq \dim \text{span}(T(e_1), \dots, T(e_n), b^T) = \dim \text{span}(C_1, C_2, \dots, C_n, b) = \text{rank}(A\vdots b).$$

since $\text{span}(T(e_1), \dots, T(e_n)) \subseteq \text{span}(T(e_1), \dots, T(e_n), b)$.

We know that $\text{span}(T(e_1), \dots, T(e_n)) = \text{span}(T(e_1), \dots, T(e_n), b^T)$

$\iff b^T \in \text{span}(T(e_1), \dots, T(e_n)) = \text{Im}\, T \iff \exists z \in F^n : T(z) = b^T$

$\iff$ The system has a solution z $\iff$ consistent.

The system is dependent $\iff \exists z_1, z_2 : z_1 \neq z_2,\ \begin{matrix} Az_1 = b \\ Az_2 = b \end{matrix}$

$\iff \exists w = z_1 - z_2 \neq 0 : Aw = A(z_1 - z_2) = 0$

$\iff \exists w^T \neq 0 \in F^n : T(w^T) = 0$

$\iff T$ is not a monomorphism

$\iff \dim \ker T \geq 1$ and $\dim F^n = \dim \ker T + \dim \text{Im}\, T > \dim \text{Im}\, T$

$> \text{rank } A.$

$\blacksquare$

Made with Goodnotes

Corollary 1 : $A \in M_{m\times n}(F)$, $Ax = 0$

The system $Ax = 0$ is always consistent

and dependent $\iff$ rank $A < n$

independent $\iff$ rank $A = n$

Proof: It is clear that rank A = rank $(A \,;\, 0)$ since span $(C_1, \dots, C_n)$ = span $(C_1, \dots, C_n, 0)$

Corollary 2 : $A \in M_{m\times n}(F)$, $Ax = 0$

If $m < n \Rightarrow Ax = 0$ is consistent dependent.

Proof: $Ax = 0$ consistent independent $\iff$ rank $A = n$

since $A \in M_{m\times n}(F) \Rightarrow n =$ rank A = column rank A

$= \dim \operatorname{span}(C_1, \dots, C_n) \leq \dim F^m = m$

Invertible matrices

Def: $A \in M_{m\times n}(F)$

1) We say that A has a left inverse if $\exists\, B \in M_{n\times m}(F)$: $B \cdot A = Id_n$

2) " " Right " $\exists\, C \in M_{n\times m}(F)$: $A \cdot C = Id_m$

3) " " An inverse " $\exists\, B \in M_{n\times m}(F)$: $B \cdot A = Id_n$ $A \cdot B = Id_m$.

Lemma 1 : $A \in M_{m\times n}(F)$. If A has left inverse $B \in M_{n\times m}$ and right inverse $C \in M_{n\times m}$ then $B = C$.

Proof : $\underset{n\times m}{B} = B \cdot Id_m = B \cdot (A \cdot C) = (B \cdot A) \cdot C = Id_n \cdot \underset{n\times m}{C} = C$.

Lemma 2 : $A \in M_{m\times n}(F)$ and it has inverse, then it is unique

Proof : Assume $B_1 A = Id_n$, $AB_1 = Id_m$; $B_2 A = Id_n$, $AB_2 = Id_m$ $\Longrightarrow$ B_1 is a left inverse and B_2 is a right inverse $\underset{\text{lemma 1}}{\Longrightarrow}$ $B_1 = B_2$

∎

Theorem : $A \in M_n(F)$. Following are equivalent

1) A has left inverse
2) rank $A = n$
3) A is row equivalent to Id
4) A is a product of elementary matrices
5) A has inverse.

5.12. Lecture.

Theorem: $A, B \in M_n(F)$.

1) A invertible $\Leftrightarrow$ A^{-1} is invertible.

2) A, B invertible $\Leftrightarrow$ AB and BA are invertible.

Proof: 1) $A \cdot A^{-1} = Id_n = A^{-1} \cdot A$

2) Assume A and B are invertible

$A \cdot B \cdot B^{-1} \cdot A^{-1} = Id = B^{-1} \cdot A^{-1} \cdot A \cdot B$, $B \cdot A \cdot A^{-1} \cdot B^{-1} = Id = A^{-1} \cdot B^{-1} \cdot B \cdot A$.

$\blacksquare$

Theorem: $A \in M_n(F)$. Following are equivalent

1) A has left inverse

2) $\operatorname{rank} A = n$

3) A is row equivalent to Id

4) A is a product of elementary matrices

5) A has inverse.

Proof: $5 \Rightarrow 1)$ ✓

$4 \Rightarrow 5)$ $A = E_1 \cdots E_s$, E_i elementary matrices

By theo 1, (2), using induction, $A^{-1} = E_s^{-1} \cdot E_{s-1}^{-1} \cdots E_1^{-1}$

$3 \Rightarrow 4)$ $A \sim Id$ (row equivalent) $\Longrightarrow \exists E_1, E_2, \cdots, E_s$ row equivalent matrices s.t.

$E_s \cdots E_2 \cdot E_1 \cdot A = Id \Rightarrow E_s^{-1}(E_s \cdots E_2 \cdot E_1) \cdot A = E_s^{-1} \cdot Id = E_s^{-1}$

$\Rightarrow E_{s-1} \cdots E_2 \cdot E_1 \cdot A = E_s^{-1} \Rightarrow A = E_1^{-1} \cdot E_2^{-1} \cdots E_{s-1}^{-1} \cdot E_s^{-1}$

$1 \Rightarrow 2)$ Assume A has left inverse $\Longleftrightarrow \exists B \in M_n(F) : B \cdot A = Id$

Consider $Ax = 0$: we know $\{x : Ax = 0\} = \{0\} \Leftrightarrow Ax = 0$ consistent independent

$\Leftrightarrow \text{rank } A = n.$

$T: F^n \to F^n$, $T(x^T) = (Ax)^T$. $Ax = 0$ consistent independent

$\Longleftrightarrow \ker T = \{x^T : T(x^T) = 0\}$

$= \{x^T : (Ax)^T = 0\}$

$= \{x^T : Ax = 0\} = \{0\}.$

$\Rightarrow n = \dim \ker T + \dim \operatorname{Im} T. = 0 + \dim \operatorname{Im} T.$

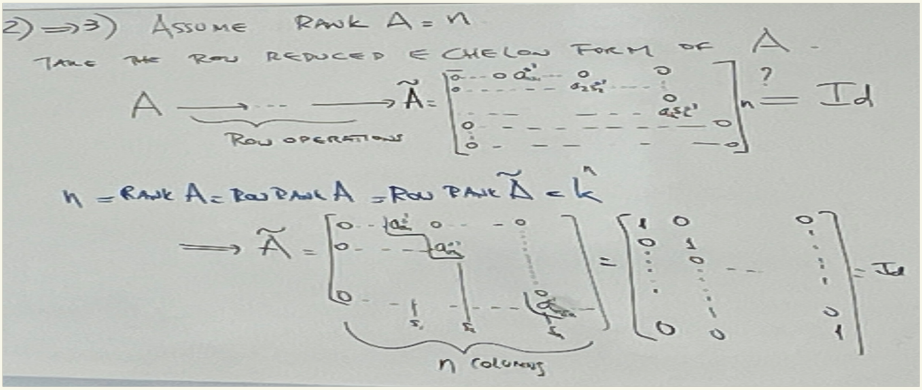
Now, $\operatorname{Im} T = \operatorname{span}(T(e_1), \dots, T(e_n)) = \operatorname{span}((Ae_1)^T, \dots, (Ae_n)^T)$

$= \operatorname{span}(c_1^T, \dots, c_n^T)$

$\dim T = \dim \operatorname{span}(c_1, \dots, c_n) = \text{column rank } A = \text{rank } A.$

We have proven : $\{x : Ax = 0\} = \{0\} \Rightarrow \text{rank } A = n$

If $BA = Id$ and $Ax = 0 \Rightarrow B.(Ax) = B \cdot 0 \Rightarrow x = 0 \Rightarrow \text{rank } A = n$



∎

Remark : If $A \in M_{m \times n}(F)$ has left and right inverse then $n = m$ and A invertible.

Proof : $\exists B \in M_{n \times m}(F)$, $\exists C \in M_{n \times m}(F)$: $B \cdot A = Id_n$, $A \cdot C = Id_m$

$$\Rightarrow \begin{cases} B \cdot A = Id_n \xRightarrow{\text{Theo } 2,\, 1 \Rightarrow 2} \text{rank}(A) = n \Rightarrow n = \text{rank } A = \text{row rank } A \le m \\ A \cdot C = Id_m \xRightarrow{\text{Theo } 2,\, 1 \Rightarrow 2} \text{rank}(C) = m \Rightarrow m = \text{rank } C = \text{row rank } C \le n \end{cases} \Rightarrow m = n$$

∎

WE HAVE CONNECTED SYSTEMS OF LINEAR EQUATIONS WITH LINEAR MAPS

$Ax=b \longrightarrow T: F^n \to F^m$, $T(x_1, \dots, x_n) = (a_{11}x_1 + \dots + a_{1n}x_n, \dots, a_{m1}x_1 + \dots + a_{mn}x_n) = (b_1, \dots, b_m)$

$T: F^n \to F^m$, T IS UNIQUELY DETERMINED BY $T(e_1), T(e_2), \dots, T(e_n) \in F^m$

$Ax=b \rightsquigarrow$

$T(e_1) = (a_{11}, a_{21}, \dots, a_{m1})$
$T(e_2) = (a_{12}, a_{22}, \dots, a_{m2})$
$\vdots$
$T(e_n) = (a_{1n}, a_{2n}, \dots, a_{mn})$

$\longrightarrow T(x_1, x_2, \dots, x_n) = x_1 T(e_1) + \dots + x_n T(e_n)$

NOW, WE WANT TO CONNECT LINEAR MAPS WITH MATRICES

Def: $T: V \to W$ linear map between two F-vector spaces. Let B_1 basis of V, B_2 basis of W. $B_1 = \{v_1, \dots, v_n\}$, $B_2 = \{w_1, \dots, w_m\}$.

The matrix of T with respect to B_1 and B_2 is defined by:

$$[T]_{B_1 B_2} = M(T, B_1, B_2) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in M_{m \times n}(F) \text{ s.t. } T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \dots + a_{mj}w_m = \sum_{i=1}^{m} a_{ij} w_i$$

EXAMPLE: $T: \mathbb{R}^2 \to \mathbb{R}^3$, $T(x,y) = (2x+y, x, 3y+x)$

$\mathcal{C}_2 = \{(1,0), (0,1)\}$ $\quad \mathcal{C}_3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

$B_1 = \{(1,0), (1,1)\}$ $\quad B_2 = \{(1,0,0), (0,1,1), (0,-1,1)\}$

$$[T]_{\mathcal{C}_2 \mathcal{C}_3} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}$$

$T(1,0) = (2,1,1) = 2(1,0,0) + 1(0,1,0) + 1(0,0,1)$
$T(0,1) = (1,0,3) = 1(1,0,0) + 0(0,1,0) + 3(0,0,1)$

$$[T]_{B_1 B_2} = \begin{bmatrix} 2 & 3 \\ 1 & 5/2 \\ 0 & 3/2 \end{bmatrix}$$

$T(1,0) = (2,1,1) = 2(1,0,0) + 1(0,1,1) + 0(0,-1,1)$
$T(1,1) = (3,1,4) = 3(1,0,0) + \frac{5}{2}(0,1,1) + \frac{3}{2}(0,-1,1)$

Theorem:

$T, T': V \to W$, B_1 basis for V, B_2 basis for W. $\lambda \in F$.

1). $[T+T']_{B_1 B_2} = [T]_{B_1 B_2} + [T']_{B_1 B_2}$ $\quad$ 2) $[\lambda T]_{B_1 B_2} = \lambda [T]_{B_1 B_2}$.

$T: V \to W$, $T': W \to U$, B_3 basis for U.

$[T' \circ T]_{B_1 B_3} = [T']_{B_2 B_3} \cdot [T]_{B_1 B_2}$

Proof: 1). $B_1 = \{v_1, \dots, v_n\}$ $B_2 = \{w_1, \dots, w_n\}$

$$[T+T']_{B_1B_2} = \begin{bmatrix} [(T+T')(v_1)]_{B_2} & \cdots & [(T+T')(v_n)]_{B_2} \end{bmatrix}$$

$$(T+T')(v_j) \underset{Def}{=} T(v_j) + T'(v_j) = \sum a_{ij} w_i + \sum a'_{ij} w_i = \sum (a_{ij} + a'_{ij}) w_i$$

where $[T]_{B_1B_2} = [a_{ij}]$ and $[T']_{B_1B_2} = [a'_{ij}]$

$$\Rightarrow [T+T']_{B_1B_2} = [a_{ij} + a'_{ij}] = [a_{ij}] + [a'_{ij}] = [T]_{B_1B_2} + [T']_{B_1B_2}.$$

2). $$[\lambda T]_{B_1B_2} = \begin{bmatrix} [(\lambda T)(v_1)]_{B_2} & \cdots & [(\lambda T)(v_n)]_{B_2} \end{bmatrix}$$

$$(\lambda T)(v_j) = \lambda \cdot T(v_j) = \lambda \cdot \sum_{i=1}^{m} a_{ij} w_i = \sum_{i=1}^{m} (\lambda a_{ij}) w_i$$

$$[\lambda T]_{B_1B_2} = [\lambda a_{ij}] = \lambda [a_{ij}] = \lambda \cdot [T]_{B_1B_2}$$

3). $$\underset{B_1}{V} \xrightarrow{T} \underset{B_2}{W} \xrightarrow{T'} \underset{B_3}{U}$$

$$[T]_{B_1B_2} = [a_{ij}] \iff T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \forall j = 1, \dots, n.$$

$$[T']_{B_2B_3} = [b_{st}] \iff T'(w_t) = \sum_{s=1}^{k} b_{st} u_s \quad \forall t = 1, \dots, m$$

$$(T' \circ T)(v_j) = T'(T(v_j)) = T'\left(\sum_{i=1}^{m} a_{ij} w_i\right) = \sum_{i=1}^{m} a_{ij} T'(w_i) = \sum_{i=1}^{m} \underset{\in F}{a_{ij}} \cdot \sum_{s=1}^{k} \underset{\in F}{b_{si}} \underset{\in W}{u_s}$$

$$= \sum_{i=1}^{m} \sum_{s=1}^{k} b_{si} \cdot a_{ij} \cdot u_s = \sum_{s=1}^{k} \left(\sum_{i=1}^{m} b_{si} a_{ij} \right) \cdot u_s.$$

$$[T' \circ T]_{B_1B_3} = \left[\left(\sum_{i=1}^{m} b_{si} a_{ij} \right)_{sj} \right] = [b_{st}] \cdot [a_{ij}] = [T']_{B_2B_3} \cdot [T]_{B_1B_2}.$$

▨

EXAMPLE: $T(x,y) = (x+y,\ x-y)$, $\quad T'(x,y) = (2x,\ x-y,\ 3y)$

$$\underset{\mathcal{C}_2}{\mathbb{R}^2} \xrightarrow{\ T\ } \underset{\mathcal{C}_2}{\mathbb{R}^2} \xrightarrow{\ T'\ } \underset{\mathcal{C}_3}{\mathbb{R}^3}$$

$$(T' \circ T)(x,y) = T'(T(x,y)) = T'(x+y,\ x-y) = \big(2(x+y),\ (x+y)-(x-y),\ 3(x-y)\big) = (2x+2y,\ 2y,\ 3x-3y)$$

$$[T' \circ T]_{\mathcal{C}_2\mathcal{C}_3} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 3 & -3 \end{bmatrix}$$

$$(T' \circ T)(1,0) = (2,0,3) = 2(1,0,0) + 0(0,1,0) + 3(0,0,1)$$

$$(T' \circ T)(0,1) = (2,2,-3) = 2(1,0,0) + 2(0,1,0) + (-3)(0,0,1)$$

$$\begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \\ 3 & -3 \end{bmatrix}$$

$$[T']_{\mathcal{C}_2\mathcal{C}_3} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 3 \end{bmatrix}$$

$T'(1,0) = (2,1,0)$

$T'(0,1) = (0,-1,3)$

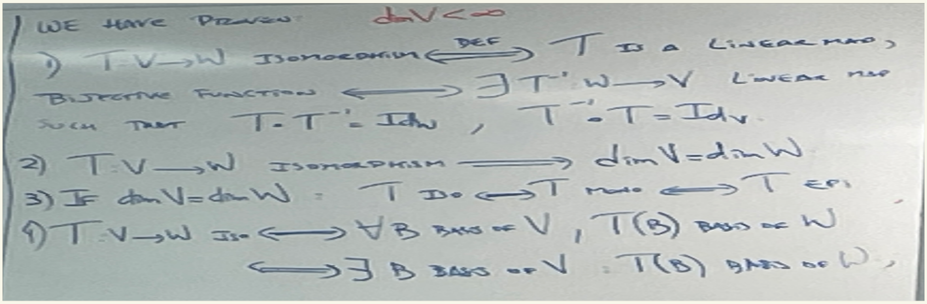
$$[T]_{\mathcal{C}_2\mathcal{C}_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$T(1,0) = (1,1)$

$T(0,1) = (1,-1)$

$$[T']_{\mathcal{C}_2\mathcal{C}_3} \cdot [T]_{\mathcal{C}_2\mathcal{C}_2} = [T' \circ T]_{\mathcal{C}_2\mathcal{C}_3}$$

5.14 Lecture



Def: Let V and W be two $\mathbb{F}$-vector spaces. We say that V and W are isomorphic if $\exists\, T: V \to W$, T an isomorphism.

Remark: "Being isomorphic" is an equivalence relation

$$V \sim W \iff \exists\, T: V \to W \text{ isomorphism}$$

i) $V \sim V \iff id: V \to V$ iso ✓

ii) If $V \sim W \Rightarrow \exists\, T: V \to W$ iso $\Rightarrow \exists\, T^{-1}: W \to V$ iso $\Rightarrow W \sim V$.

iii) If $V \sim W$ and $W \sim U \Rightarrow \exists\, T_1: V \to W$, $T_2: W \to U$ iso $\Rightarrow \exists\, T_2 \circ T_1: V \to U$ iso

$\Rightarrow V \sim U$. ✓

Theorem: Let V, W two $\mathbb{F}$-vector spaces, $V \sim W \iff \dim V = \dim W$.

Proof: $\Rightarrow$) We know that $\exists\, T: V \to W$ iso. Take B a basis of V. By (4) we know $T(B)$ is a basis for W.

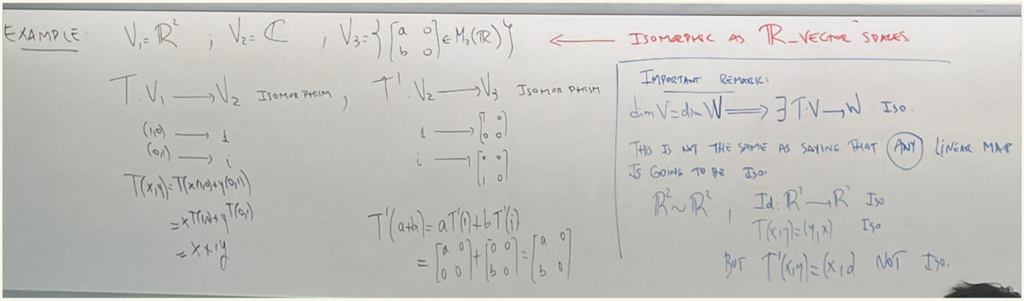
$$\Rightarrow \dim V = \#B \underset{T \text{ injective}}{=\!=} \#T(B) = \dim W$$

$\Leftarrow$) Assume $\dim V = \dim W = \#I$, take $B = \{v_i, i \in I\}$ a basis of V
$B' = \{w_i, i \in I\}$ a basis of W.

Define $T: V \to W$ s.t. $T(v_i) = w_i$

$\Rightarrow T(\sum a_i v_i) = \sum a_i w_i \quad \Rightarrow T(B) = B'$ By (¥). T is iso. ✓

∎



Theorem: Let V, W be two F-vector spaces, $\dim V = n$, $\dim W = m$.

$\text{Hom}_F(V, W)$ is isomorphic to $M_{m\times n}(F)$.

Proof: $\alpha: \text{Hom}_F(V, W) \longrightarrow M_{m\times n}(F)$, $\alpha(T) = [T]_{B_1 B_2}$. B_1 is a basis for V, B_2 a basis for W.

$\alpha(T_1 + T_2) = [T_1 + T_2]_{B_1 B_2} = [T_1]_{B_1 B_2} + [T_2]_{B_1 B_2} = \alpha(T_1) + \alpha(T_2)$.

$\alpha(\lambda \cdot T) = [\lambda T]_{B_1 B_2} = \lambda \cdot [T]_{B_1 B_2} = \lambda \cdot \alpha(T)$

mono: $\alpha(T) = 0 \Rightarrow T(v_j) = 0 \;\; \forall v_j \in B_1 \Rightarrow T = 0$.

epi: $A = [a_{ij}]$ Define $T(v_j) = \sum a_{ij} w_i \Rightarrow \alpha(T) = A$.

∎

Corollary: $\dim \text{Hom}_F(V, W) = \dim V \cdot \dim W = n \cdot m$

Definition: Let V be an F-vector space, $B = \{v_1, \cdots, v_n\}$ a basis for V. We define the matrix of v with respect to B as follows: (55).

$$[v]_B \in M_{n\times 1}(F), \quad [v]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ if } v = a_1 v_1 + \cdots + a_n v_n .$$

Example: $V = \mathbb{R}^3$, $v = (2,1,0)$, $B_3 = \{e_1, e_2, e_3\}$. $B = \{(1,0,0), (0,1,1), (1,0,1)\}$.

$$[v]_{B_3} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad (2,1,0) = 2(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

$$[v]_B = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \qquad (2,1,0) = 3(1,0,0) + 1(0,1,1) + -1(1,0,1)$$

Theorem: Let V be an F-vector space, $\dim V = n$. Then V is isomorphic to $M_{n\times 1}(F)$. and the map $V \xrightarrow{\beta} M_{n\times 1}(F)$, $v \longrightarrow [v]_B$ for B a basis for V. is an isomorphism

Proof: Let $B = \{v_1, \cdots, v_n\}$ $\beta(v) = [v]_B$, β is a linear map

$(v = \sum a_i v_i, \ u = \sum b_i v_i)$.

$$\beta(\lambda v) = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix} = \lambda \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \lambda \cdot \beta(v).$$

$$\beta(v+u) = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \beta(v) + \beta(u)$$

mono: $v \in \ker \beta \Rightarrow \beta(v) = [v]_B = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Longrightarrow v = \sum 0 \cdot v_i = 0.$ ✓.

epi: $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in M_{n\times 1}(F)$, take $v = b_1 v_1 + \cdots + b_n v_n \Longrightarrow \beta(v) = [v]_B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ ✓

▨

Theorem : $[T(v)]_{B'} = [T]_{BB'} \cdot [v]_B$

$$\underset{M_{m\times 1}(F)}{\cap} \qquad \underset{M_{m\times n}(F)}{\cap} \qquad \underset{M_{n\times 1}(F)}{\cap}$$

Proof : $B = \{v_1, \cdots, v_n\}$ $\quad [v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $v = a_1 v_1 + \cdots + a_n v_n$

$B' = \{w_1, \cdots, w_m\}$

$$[T]_{BB'} = \begin{bmatrix} b_{11} & \cdots & b_{1j} & b_{1m} \\ b_{m1} & \cdots & b_{mj} & b_{mn} \end{bmatrix} \qquad T(v_j) = b_{1j} w_1 + \cdots + b_{mj} \cdot w_m = \sum_{i=0}^{m} b_{ij} w_i$$

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n)$$

$$= a_1 \sum_{j=1}^{m} b_{i1} w_i + \cdots + a_n \sum_{i=1}^{m} b_{in} w_i$$

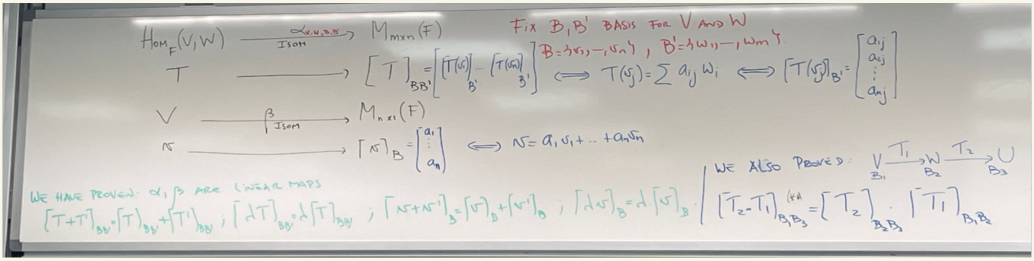
$$= \sum_{j=1}^{n} \sum_{i=1}^{m} a_j b_{ij} w_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} b_{ij} a_j \right) w_i$$

$$\Rightarrow [T(v)]_{B'} = \begin{bmatrix} \sum_{j=1}^{n} b_{1j} a_j \\ \vdots \\ \sum_{j=1}^{n} b_{mj} a_j \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \checkmark$$

$\blacksquare$

5.19 Lecture.



Theorem : $T \in Hom_F(V, W)$, $dim_F V = n$, $dim_F W = n$

Then 1) : T is an isomorphism $\quad$ 2) : $[T]_{BB'}$ is invertible $\forall B, B'$ basis of V and W

$\iff$ 3). $[T]_{BB'}$ invertible for some B, B' basis of V and W.

Proof : $1 \Rightarrow 2)$: We know that $T : V \to W$ is an isomorphism $\overset{\text{Theo}}{\iff}$ $\exists T^{-1} : W \to V$,

$T^{-1} \in Hom_F(W, V)$ s.t. $T^{-1} \circ T = Id_V$, $T \circ T^{-1} = Id_W$.

Let B, B' basis of V and W

$$[T^{-1}]_{B'B} \cdot [T]_{BB'} = [T^{-1} \circ T]_{BB} = [Id_V]_{BB} = [[Id v_1]_B \cdots [Id v_n]_B] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ 0 & 0 & \ddots & \\ 0 & & & 1 \end{bmatrix} = Id_n$$

In the same way : $[T]_{BB'} \cdot [T^{-1}]_{B'B} = [T \circ T^{-1}]_{B'B'} = [Id_W]_{B'} = Id_n$

$\Rightarrow Id_n = [T]_{BB'} \cdot [T^{-1}]_{B'B} = [T^{-1}]_{B'B} \cdot [T]_{BB'}$

$\Rightarrow [T]_{BB'}$ is invertible.

$2 \Rightarrow 3)$ ✓ $\forall (B, B') \Rightarrow$ True for a particular pair (B, B').

$3 \Rightarrow 1)$ We know that $\exists B, B' : [T]_{BB'}$ invertible. Let $C = [T]_{BB'} \in M_n(F)$.

$\Rightarrow \exists C^{-1} \in M_n(F) : C \cdot C^{-1} = Id_n = C^{-1} \cdot C$.

Using that α is an isomorphism we know that $\exists S \in Hom_F(W, V) : \alpha(S) = [S]_{B'B} = C^{-1}$

$\alpha_{W,V} : Hom_F(W, V) \longrightarrow M_{n\times n}(F)$

$S \longrightarrow [S]_{B'B} = C^{-1}$

$\alpha_{VV} : \underset{B\,B}{Hom(V,V)} \rightarrow M_n(F)$

$\alpha_{WW} : \underset{B'\,B'}{Hom(W,W)} \longrightarrow M_n(F)$

$\underset{B}{V} \xrightarrow{T} \underset{B'}{W} \xrightarrow{S} \underset{B}{V}$

Let's see that $S = T^{-1}$

$[S\circ T]_{BB} = [S]_{B'B} \cdot [T]_{BB'} = C^{-1}\cdot C = Id_n = [Id_V]_{BB}$

$\Rightarrow \alpha_{V,V}(S\circ T) = \alpha_{V,V}(Id_V) \xrightarrow{\alpha \text{ iso}} S\circ T = Id_V.$

$[T\circ S]_{B'B'} = [T]_{BB'} \cdot [S]_{B'B} = C\cdot C^{-1} = Id_n = [Id_W]_{B'B'}$

$\Rightarrow \alpha_{W,W}(T\circ S) = \alpha_{W,W}(Id_W) \Longrightarrow T\circ S = Id_W.$

∎

Change of Basis

V. B_1, B_2 Basis : $[v]_{B_1} \xrightarrow{?} [v]_{B_2}$

$\underset{B_1, B_2}{V} \xrightarrow{T} \underset{B_1', B_2'}{W}$ $\quad [T]_{B_1B_1'} \xrightarrow{?} [T]_{B_2B_2'}$

Def: Let V be an F-vector space, B_1, B_2 two basis. The change of basis matrix from B_1 to B_2 is the matrix

$$M(Id, B_1, B_2) = C(B_1, B_2) = [B_1]_{B_2} \; = [Id_V]_{B_1B_2} = \alpha(Id_V).$$

Theorem: V, W F-vector spaces, B_1, B_2 basis for V, B_1', B_2' basis for W. $T \in Hom_F(V, W)$

1) $[B_1]_{B_2} \cdot [B_2]_{B_1} = Id_n$ that is $[B_1]_{B_2}^{-1} = [B_2]_{B_1}$

2) $[B_1]_{B_2} [v]_{B_1} = [v]_{B_2}$

3) $[B_1']_{B_2'} [T]_{B_1B_1'} [B_2]_{B_1} = [T]_{B_2B_2'}$

Proof:

1) $[B_1]_{B_2} \cdot [B_2]_{B_1} = [Id_V]_{B_1B_2} \cdot [Id_V]_{B_2B_1} = [Id_V]_{B_2B_2} = Id_n$

2) $[B_1]_{B_2} \cdot [v]_{B_1} = [Id_V]_{B_1B_2} \cdot [v]_{B_1} = [Id_V(v)]_{B_2} = [v]_{B_2}$

3) $[B_1']_{B_2'} \cdot [T]_{B_1B_1'} \cdot [B_2]_{B_1} = [Id_W]_{B_1'B_2'} \cdot [T]_{B_1B_1'} \cdot [Id_V]_{B_2B_1} = [Id_W \circ T]_{B_1B_2'} \cdot [Id_V]_{B_2B_1}$

$= [T]_{B_2B_2'}$

EXAMPLE 1. $T: \mathbb{R}^2 \to \mathbb{R}^3$, $[T]_{\mathcal{E}_2\mathcal{E}_3} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$

$$[T(x,y)]_{\mathcal{E}_3} \overset{(**)}{=} [T]_{\mathcal{E}_2\mathcal{E}_3} \cdot [(x,y)]_{\mathcal{E}_2} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$(x,y) = x\cdot(1,0) + y\cdot(0,1)$

$$= \begin{bmatrix} x-y \\ 2y \\ x+3y \end{bmatrix}$$

$T(x,y) = (x-y)\cdot(1,0,0) + 2y\cdot(0,1,0) + (x+3y)(0,0,1)$

$= (x-y, 2y, x+3y)$.

EXAMPLE 2. DESCRIBE $T: \mathbb{R}^2 \to \mathbb{R}^3$ SUCH THAT

$[T]_{B_1B_2} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$, $B_1 = \{(1,0), (1,1)\}$, $B_2 = \{(1,0,0), (1,0,1), (0,1,1)\}$

OPTION 1. $[T(x,y)]_{B_2} = [T]_{B_1B_2} \cdot [(x,y)]_{B_1} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-y \\ y \end{bmatrix}$

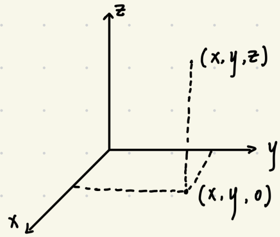
$(x,y) = (x-y)(1,0) + y\cdot(1,1)$

$$= \begin{bmatrix} x-2y \\ 2x-y \\ x-y \end{bmatrix}$$

$T(x,y) = (x-2y)(1,0,0) + (2x-y)(1,0,1) + (x-y)(0,1,1)$

$= (3x-3y, x-y, 3x-2y)$

Projection



Def: A projection is a linear map $P: V \to V$ s.t. $P \circ P = P$

Lemma: P is a projection $\iff p(w) = w \quad \forall w \in \operatorname{Im} p$

Proof: $\Rightarrow)$ $w \in \operatorname{Im} p \Rightarrow w = p(v) \Rightarrow p(w) = p(p(v)) = p(v) = w$.

$\Leftarrow)$ $(p \circ p)(v) = p(\underbrace{p(v)}_{w \in \operatorname{Im} p}) = p(v)$ ✓

Theorem: $P: V \to V$ linear map

i) $P \circ P = P \Rightarrow V = \ker p \oplus \operatorname{Im} p$

ii) If $V = S \oplus U$, then $\exists!\ p: V \to V$ s.t. $\ker p = S$ and $\operatorname{Im} p = U$.

Proof: 1) $\ker p \subseteq V$. $\operatorname{Im} p \subseteq V$ subspaces $\Rightarrow \ker p + \operatorname{Im} p \subseteq V$

Let $v \in V \Rightarrow v = \underbrace{p(v)}_{\in \operatorname{Im} p} + \underbrace{v - p(v)}_{\in \ker p}$ and $p(v - p(v)) = p(v) - (p \circ p)(v) = p(v) - p(v) = 0$

$\Rightarrow V \subseteq \ker p + \operatorname{Im} p \quad \Rightarrow V = \ker p + \operatorname{Im} p$.

Let $w \in \ker p \cap \operatorname{Im} p \Rightarrow p(w) = 0$ and $w = p(v)$, $v \in V$

then $\underset{w \in \ker p}{0 = p(w)} \underset{w \in \operatorname{Im} p}{=} p(p(v)) = (p \circ p)(v) \underset{p \text{ projection.}}{=} p(v) = w$

$\Rightarrow \ker p \cap \operatorname{Im} p = \{0\}$

2) We have to define $p: V \to V$ such that: $\begin{cases} p \text{ is a linear map} \\ p \text{ is a projection} \\ \ker p = S \\ \operatorname{Im} p = U. \end{cases}$

$p(v) = p(s+u) = p(s) + p(u) = 0 + u \Rightarrow p$ is unique.

P is a linear map:

$p(v_1+v_2) = p(s_1+u_1+s_2+u_2) = p(\underbrace{s_1+s_2}_{\in S}+\underbrace{u_1+u_2}_{\in U}) = u_1+u_2 = p(v_1)+p(v_2).$ ✓

$p(\lambda \cdot v) = p(\lambda(s+u)) = p(\underbrace{\lambda s}_{\in S}+\underbrace{\lambda u}_{\in U}) = \lambda u = \lambda \cdot p(v).$ ✓

P is a projection:

$(p \circ p)(v) = (p \circ p)(s+u) = p(p(s+u)) = p(u) = p(0+u) = u = p(v)\,,\ \forall v \Rightarrow p \circ p = p$

$\ker p = \{v : p(v) = 0\} = \{v = s+u : p(v) = p(s+u) = u = 0\} = \{v = s+u : u = 0\} = S.$

$\operatorname{Im} p = \{p(v), v \in V\} = \{p(v) = p(s+u) = u\,, v \in V\} = U.$

∎

5.21 Lecture.

Corollary: Let $p: V \to V$ be a projection. Then $\exists B$ a basis of V such that:

$$[p]_B = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{bmatrix} = \left[\begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array}\right] \in M_n(F) \quad \text{if } \dim V = n$$

Proof: We know $V = \ker p \oplus \operatorname{Im} p$. Take $\{v_1, \cdots, v_r\}$ a basis for $\operatorname{Im} p$ and $\{v_{r+1}, \cdots v_n\}$ a basis for $\ker p \Rightarrow B = \{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$ is a basis of V.

$$[p]_B = \Big[[p(v_1)]_B \; [p(v_2)]_B \cdots [p(v_r)]_B \; \underset{0}{[p(v_{r+1})]_B} \cdots \underset{0}{[p(v_n)]_B} \Big] = \Big[[v_1]_B \cdots [v_r]_B \; 0 \cdots 0 \Big]$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 1 & \vdots & & \vdots \\ 0 & \cdots & \cdots & 1 & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$\blacksquare$

Remark: 1) The previous corollary can be generalized as follows:

p is a projection $\iff \exists B$ a basis s.t. $[p]_B = \left[\begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array}\right]$

2). The previous corollary can be generalized to any linear map:

$T: V \to W$, but we need two basis.

Theorem: Let $T: V \to W$ a linear map, $\dim \operatorname{Im} T = r$. Then $\exists B$ a basis for V, and B' a basis for W s.t. $[T]_{BB'} = \left[\begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array}\right] \in M_{m \times n}(F)$

if $\dim V = n$. $\dim W = m$.

Proof: since $\dim \operatorname{Im} T = r$, we know $\dim \ker T = \dim V - \dim \operatorname{Im} T = n-r$

Take $\{v_{r+1}, \cdots, v_n\}$ a basis for $\ker T$.

$\{v_{r+1}, \cdots, v_n\}$ is L.I. in V, we can extend it to a basis of V

$B = \{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$.

We know $\operatorname{Im} T = \{T(v), v \in V\} = \{T(\sum_{i=1}^{n} a_i v_i) = a_1 T(v_1) + \cdots + a_r T(v_r) + a_{r+1} \underset{0}{T(v_{r+1})} + \cdots + a_n \underset{0}{T(v_n)}\}$

$= \{a_1 T(v_1) + \cdots + a_r T(v_r)\} = \operatorname{span}\{T(v_1), \cdots, T(v_r)\}$

and $\dim \operatorname{Im} T = r$

$\Rightarrow \{w_1 = T(v_1), w_2 = T(v_2), \cdots, w_r = T(v_r)\}$ is a basis for $\operatorname{Im} T$.

$\Rightarrow \{w_1, \cdots, w_r\}$ is L.I. in W, we can extend it to a basis

$B' = \{w_1, \cdots, w_r, w_{r+1}, \cdots, w_n\}$ a basis for W.

$$[T]_{BB'} = \left[[T(v_1)]_{B'} \cdots [T(v_r)]_{B'} \ [T(v_{r+1})]_{B'} \cdots [T(v_n)]_{B'} \right] = \left[[w_1]_{B'} - [w_r]_{B'} \ 0 \cdots 0 \right]$$

$$= \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

∎

Determinant only for square matrix

Def: $n=1 \quad A=(a) \quad |A| = a$

$n=2 \quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad |A| = ad-bc$

$n=3 \quad A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

example

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\underbrace{\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}}_{M_{12}} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$= 1(45-48) - 2(36-42) + 3(32-35)$

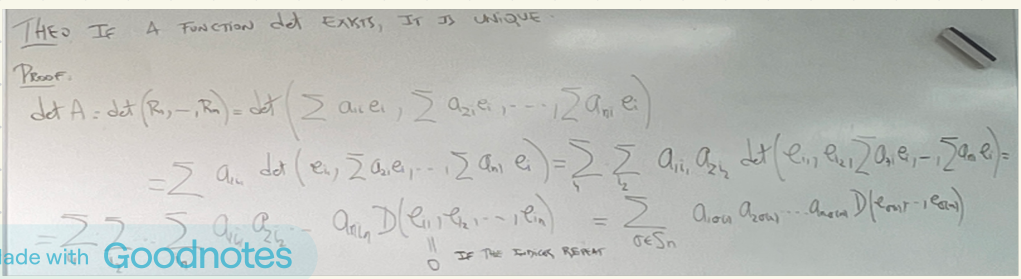
$= -3 + 12 - 9$

$= 0$

Def: The determinant of a matrix obtained by eliminating the i'th row and the j'th column of A is called the ij'th minor of A, and denoted M_{ij}.

Def: The determinant of $A_{n\times n}$, denoted by $|A|$ is

$$|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - \cdots (-1)^{n+1}a_{1n}M_{1n}.$$



$$= \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \, a_{1\sigma(1)} \cdots a_{n\sigma(n)} \underbrace{D(e_1, e_2, \dots, e_n)}_{1} = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

SO IT IS UNIQUE.

THEO (EXISTENCE)

THE FUNCTION $\det : M_n(F) \to F$ SATISFYING MULTILINEAR, ALTERNATING AND $\det \mathrm{Id} = 1$, EXISTS.

PROOF. BY INDUCTION:

$n=1$. $\det : M_1(F) \to F$, $\det[a] = a$ IS A DETERMINANT FUNCTION: LINEAR, ALTERNATING, $\det(1) = 1$.

ASSUMING $\det_{n-1} : M_{n-1}(F) \to F$ EXISTS, WE CAN CHECK THAT:

$$\det_n A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det_{n-1} A^{(i,j)}$$

FOR SOME FIXED i, WHERE $A^{(i,j)} = A$ WITHOUT THE ROW AND THE COLUMN WHERE a_{ij} BELONGS.

THIS FORMULA SATISFIES MULTILINEAR, ALTERNATING, $\det_n(\mathrm{Id}) = 1$

5.26 Lecture.

Def: $\det: M_n(F) \to F$ is a multilinear alterning function s.t. $\det(Id) = \det(e_1, \cdots e_n)$.

multilinear:

·) $\det(R_1, \cdots, R_i + R_i', \cdots, R_n) = \det(R_1, \cdots, R_i, \cdots, R_n) + \det(R_1, \cdots, R_i' \cdots, R_n)$.

··) $\det(R_1, \cdots, aR_i, \cdots R_n) = a \det(R_1, \cdots, R_i, \cdots, R_n)$

↑ a common factor of a row/column.

alterning:

·) $\det(R_1, \cdots, R_i, \cdots R_j, \cdots, R_n) = -\det(R_1, \cdots R_j, \cdots, R_i, \cdots, R_n)$. $i \neq j$

··) $\det(R_1, \cdots, R_i, \cdots R_i, \cdots, R_n) = 0$ or column.

Theorem: If a function det exists, then it is unique.

Proof: If $D: M_n(F) \to F$ multilinear, alterning. $D(Id) = 1$ Then.

$$D\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = D((a_{11}, \cdots, a_{1n}), \cdots (a_{m1}, \cdots, a_{mn}))$$

$$= D\left(\sum_{i_1=1}^{n} a_{1i_1} e_{i_1}, \sum_{i_2=1}^{n} a_{2i_2} e_{i_2}, \cdots, \sum_{i_n=1}^{n} a_{ni_n} e_{i_n}\right).$$

$$= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} a_{1i_1} a_{2i_2} \cdots a_{ni_n} D(e_{i_1}, e_{i_2}, \cdots, e_{i_n}).$$

$$\underset{(\cdot\cdot)}{=} \sum_{r \in S_n} a_{1r(1)} a_{2r(2)} \cdots a_{nr(n)} D(e_{r(1)}, e_{r(2)}, \cdots, e_{r(n)}).$$

$$= \sum_{r \in S_n} \operatorname{sign} r \, a_{1r(1)} \cdots a_{nr(n)} D(e_1, e_2, \cdots, e_n)$$

$$= \sum_{r} \operatorname{sign} r \, a_{1r(1)} \cdots a_{nr(n)} \Longrightarrow D \text{ is unique.}$$

▨

Theo 2: The function det exists.

Moreover, $\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A^{(i,j)} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A^{(i,j)}$

Proof: $\det_1 : M_1(F) \longrightarrow F$. $\det_1([a]) = a$ is a determinant. By induction, assume $\det_{n-1}$ exists. Define: $\varphi_1 : M_n(F) \to F$, $\varphi_2 : M_n(F) \longrightarrow F$.

$\varphi_1(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det_{n-1} A^{(i,j)}$. $\varphi_2(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det_{n-1} A^{(i,j)}$.

Check φ_1, φ_2 are multilinear, alterning, $\varphi_1(Id) = 1 = \varphi_2(Id)$.

$\Rightarrow \varphi_1$ and φ_2 are determinant function and by theo 1, $\varphi_1(A) = \varphi_2(A)$.

Theo: $A \in M_n(F)$

1). $A \xleftrightarrow[R_i \leftrightarrow R_j]{} A' \Rightarrow \det A = -\det A'$

2). $A \xrightarrow[R_i \to aR_i,\, a\neq 0]{} A' \Rightarrow \det A' = a \det A$.

3). $A \xrightarrow[R_i \to R_i + aR_j,\, a\neq 0]{} A' \Rightarrow \det A' = \det A$.

Proof: $A = \begin{pmatrix} R_1 \\ \vdots \\ R_n \end{pmatrix}$ $A' = \begin{pmatrix} R_1' \\ \vdots \\ R_n' \end{pmatrix}$

1). $\det A' = \det(R_1, \dots, \underset{i}{R_j}, \dots \underset{j}{R_i}, \dots, R_n) = (R_1, \dots, R_i, \dots R_j, \dots, R_n) = -\det A$.

2). $\det A' = \det(R_1, \dots, aR_i, \dots, R_n) = a \cdot \det(R_1, \dots, R_i, \dots, R_n) = a \det A$.

3). $\det A' = \det(R_1, \dots, R_i + aR_j, \dots, R_j, \dots, R_n)$.

$= \det(R_1, \dots, R_i, \dots, R_j, \dots, R_n) + a \cdot \det(R_1, \dots, R_j, \dots, R_j, \dots R_n)$

$= \det A + a \cdot 0 = \det A$.

▨

Theo: 1) $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & 0 & \ddots & \\ 0 & 0 & \cdots 0 & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \vdots \\ \vdots & & \ddots & 0 \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix}$.

2) $\det A = \det A^T$

3) $\det(A \cdot B) = \det A \cdot \det B$. $(\det(A+B) \neq \det A + \det B)$

4) A invertible $\Leftrightarrow \det A \neq 0$ $\qquad |A^{-1}| = \frac{1}{|A|}$

In this case, $A^{-1} = \frac{1}{\det A} \cdot \operatorname{Adj} A$. $\operatorname{Adj} A$ = Adjoint of A

Proof: 1). By induction. $|a_{11}| = a_{11}$ True for $n-1$.

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots 0 & a_{nn} \end{vmatrix} = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{vmatrix} \underset{\text{I.H.}}{=} a_{11}(a_{22} \cdot a_{33} \cdots a_{nn}).$$

2). By induction. $[a_{11}]^T = [a_{11}]$ True for $n-1$.

$$\det A = \varphi_1(A) = \sum_{i=1}^{n} (-1)^{1+j} a_{ij} \det A^{(i,j)} \underset{\text{I.H.}}{=} \sum_{i=1}^{n} (-1)^{j+1} (A^T)_{ji} \cdot \det((A^{(i,j)})^T)$$

$$= \sum_{i=1}^{n} (-1)^{j+1} (A^T)_{ji} \det(A^T)^{(j,i)}$$

$$= \varphi_2(A^T) = \det A^T.$$

3).

3) LET $D: M_n(F) \longrightarrow F$ BE GIVEN BY $D(R_1, -, R_n) = \det(R_1 B, R_2 B, \ldots, R_n B)$. $\quad [R_i] [B] = [R_i B]$

a) D IS MULTILINEAR:

$D(R_1, \ldots, R_i + R_i', \ldots, R_n) = \det(R_1 B, \ldots, \overbrace{(R_i + R_i')B}^{R_i B + R_i' B}, \ldots, R_n B)$

$= \det(R_1 B, \ldots, R_i B, \ldots, R_n B) + \det(R_1 B, \ldots, R_i' B, \ldots, R_n B) = D(R_1, -, R_i, -, R_n) + D(R_1, -, R_i', -, R_n)$.

$D(R_1, -, aR_i, -, R_n) = \det(R_1 B, \ldots, aR_i B, -, R_n B) = a D(R_1, -, R_n)$

b) D IS ALTERNATING:

$D(R_1, -, \underset{i}{R_j}, \ldots, \underset{j}{R_i}, -, R_n) = \det(R_1 B, -, R_j B, \ldots, R_i B, \ldots, R_n B) = -\det(R_1 B, \ldots, R_i B, -, R_j B, -, R_n B) = -D(R_1, -, R_i, -, R_j, -, R_n)$.

$D(R_1, -, R_1, -, R_1, -) = \det(\_\_ R_1 B, -, R_1 B, \_\_) = 0$.

c) $D(I_n) = D(e_1, e_2, -, e_n) = \det(e_1 B, e_2 B, \ldots, e_n B)$

$= \det B$ SINCE

$e_i B$ = 1-ROW OF B.

$\underset{e_i}{[0 \cdots 0\ 1\ 0 \cdots 0]} \begin{bmatrix} B \end{bmatrix} = [b_{i1}, b_{i2}, \ldots, b_{in}]$

BY THEO 1:

$D(R_1, -, R_n) = \det(R_1, -, R_n) \cdot D(e_1, -, e_n)$

$= \det(R_1, -, R_n) \cdot \det B$.

$\det(AB) = D(A) = \det A \cdot \det B$.

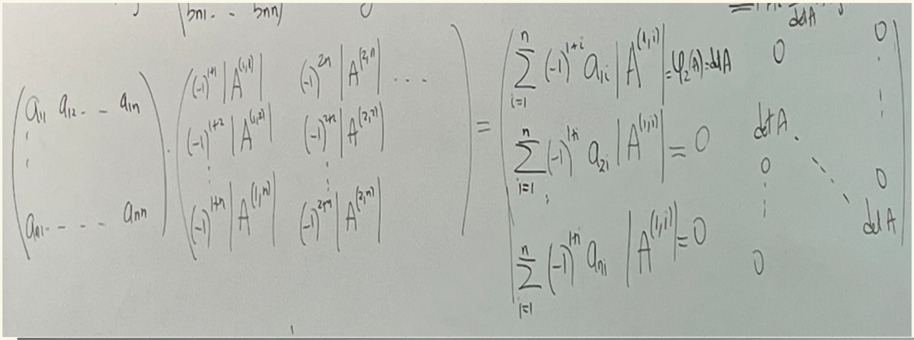
$[A] [B] = [AB]$

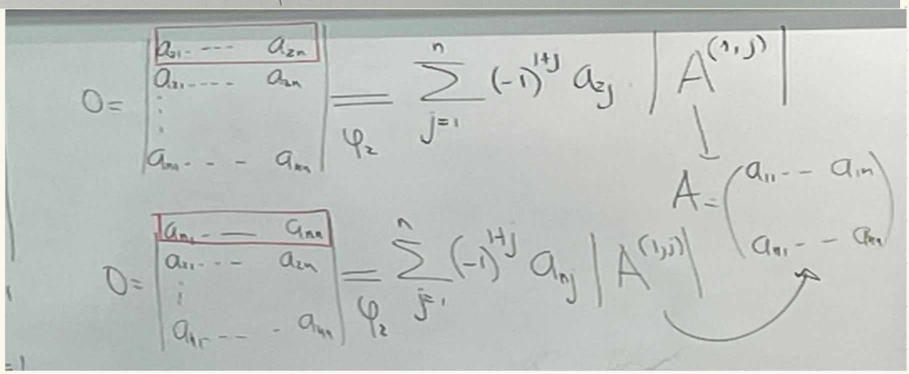
Made with Goodnotes

4). $\Rightarrow$) If A is invertible $\Rightarrow \exists A^{-1}: A \cdot A^{-1} = Id \Rightarrow 1 = \det(Id) = \det(A \cdot A^{-1})$
$= \det A \cdot \det A^{-1}$

$\Rightarrow \det A \neq 0$.

$\Leftarrow$) $Adj\ A = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$ $b_{ij} = (-1)^{i+j} \det_{n-1} A^{(j,i)}$, we will prove $A \cdot Adj\ A = \det A \cdot Id \Rightarrow A^{-1} = \frac{1}{\det A} Adj\ A$.





Corollary: every property of $|A|$ which is true for rows, is also true for columns.

Advice: Evaluate $|A|$ using a row / column with as many 0's as possible.

If one of the rows or columns is 0's then $|A| = 0$

If one row (column) is a multiple of another row (column) then $|A| = 0$.

Word of caution: $\begin{vmatrix} \alpha a & b \\ \alpha c & d \end{vmatrix} = \alpha \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

so in general: $|\alpha A| = \alpha^n |A|$ $(|\alpha A| \neq \alpha |A|)$.

If there are two same rows or colums, then $|A| = 0$.

$$A \underset{R_i \leftrightarrow R_j}{\longleftrightarrow} A' \Rightarrow \det A = -\det A' = -\det A \Rightarrow \det A = 0.$$

same row

Remarks: ① if we do row operations, $|A|$ changes!

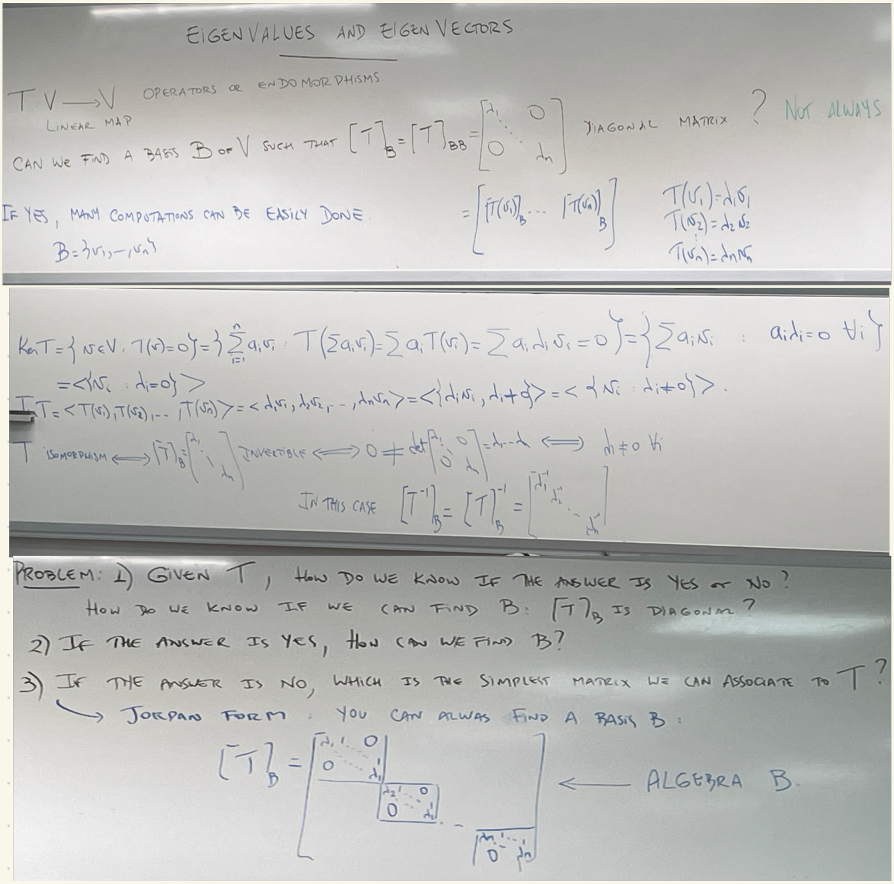
② if A and B are row-equivalent

then $|A| = 0 \Leftrightarrow |B| = 0$.

5.28 Lecture

64 – 67

Eigenvalues and Eigenvectors



$$\exists B : [T]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \iff \exists B : T(v_i) = \lambda_i v_i \quad \forall v_i \in B \iff \exists B : T(a v_i) = a T(v_i)$$

$$= a \lambda_i v_i \quad \forall v_i \in B.$$

$$\iff T(\langle v_i \rangle) \subseteq \langle v_i \rangle \quad \forall v_i \in B.$$

Def: Given $T \in \mathrm{Hom}_F(V, V)$. A subspace S of V is called Invariant under T, or T-invariant, if $T(S) \subseteq S$.

Examples:

1). $T: V \to V$, $T(0) = 0$

$$\Rightarrow T(\langle 0 \rangle) = T(\{0\}) = \{0\} \subseteq \langle 0 \rangle$$

$\langle 0 \rangle$ is T-invariant.

$T(v) \in V$, $\forall v \in V \Rightarrow T(V) \subseteq V$

V is T-invariant.

2). $O: V \to V$, $O(v) = 0 \quad \forall v \in V$.

$O(S) = \{O(v), v \in S\} = \{0\} \subseteq S \quad \forall S \subseteq V$

S is O-invariant.

3). $T: \mathbb{R}[x] \to \mathbb{R}[x]$, $T(p(x)) = p'(x)$.

$$T(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

$T(\mathbb{R}_{\leq n}[x]) = \mathbb{R}_{\leq n-1}[x] \subseteq \mathbb{R}_{\leq n}[x] \Rightarrow \mathbb{R}_{\leq n}[x]$ is T-invariant.

4). $T(x, y) = (x+y, x+y)$

$T(x, x) = 2(x, x)$

$S = \{(x, x), x \in F\}$ $\quad T(S) \subseteq S \Rightarrow S$ is T-invariant.

Problem 1:

Finding B such that $[T]_B$ is diagonal is equivalent to finding B: $\langle v_i \rangle$ is T-invariant $\forall v_i \in B$.

Definition:

Given $T \in Hom_F(V, V)$. A scalar $\lambda \in F$ is called an eigenvalue of T if $\exists v \in V, v \neq 0,$ s.t. $T(v) = \lambda v$. In this case, the non-zero vector v is called an eigenvector associated to the eigenvalue.

$E(\lambda, T) = V_\lambda = \{v \in V : T(v) = \lambda v\} = \{\text{Eigenvectors associated to } \lambda\} \cup \{0\}.$

characteristic space associated to λ.

Example: $T: F^2 \longrightarrow F^2$, $T(x, y) = (x+y, x+y)$.

Eigenvalues: $\lambda \in F$: $T(x,y) = \lambda(x,y)$ for some $(x,y) \neq 0$

$T(x,y) = \lambda(x,y) \Leftrightarrow \lambda(x,y) = (x+y, x+y) \Leftrightarrow \begin{cases} x+y = \lambda x \\ x+y = \lambda y \end{cases} \Leftrightarrow \lambda x = \lambda y$

$\Rightarrow \lambda(x-y) = 0$ — $\lambda = 0$

— $x = y \Rightarrow 2x = \lambda x. \Rightarrow x = 0$ or $\lambda = 2$

$\lambda = 0$: $T(x,y) = 0(x,y) \Leftrightarrow \begin{cases} x+y=0 \\ x+y=0 \end{cases} \Leftrightarrow x = -y.$

$T(x, -x) = (0,0) = 0 \cdot (x, -x).$

$\lambda = 0$ is an eigenvalue, and $(x, -x)$, $x \neq 0$ is an eigenvector

$V_0 = \{(x, -x), x \in F\}.$

$\lambda = 2$: $T(x,y) = 2(x,y) \Leftrightarrow \begin{cases} x+y = 2x \\ x+y = 2y \end{cases} \Leftrightarrow x = y$

$T(x,x) = (2x, 2x) = 2(x,x).$

$\lambda = 2$ is an eigenvalue, and (x,x), $x \neq 0$ is an eigenvector

$V_2 = \{(x,x), x \in F\}.$

Take $(1,-1) \in V_0$, $(1,1) \in V_2 \Rightarrow \{(1,-1),(1,1)\}$ is a basis of F^2

$T(1,-1) = 0$

$T(1,1) = 2(1,1) = 0 \cdot (1,1) + 2(1,1)$

$$[T]_B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

Proposition: If λ is an eigenvalue for a linear map $T: V \to V$, then V_λ is a subspace of V.

Proof: Option 1: Prove:

$0 \in V_\lambda$

$v_1, v_2 \in V_\lambda \Rightarrow v_1 + v_2 \in V_\lambda$

$a \in F,\ v \in V_\lambda \Rightarrow av \in V_\lambda$.

Option 2: $V_\lambda = \{v : T(v) = \lambda v\} \subseteq V$

$= \{v : T(v) = \lambda \cdot Id(v)\}$

$= \{v : (T - \lambda Id)(v) = 0\} = Ker(T - \lambda Id)$ subspace.

▨

Theorem: Let $T \in Hom_F(V,V)$. The following are equivalent:

1) $\lambda \in F$ is an eigenvalue

2) $T - \lambda Id$ is not a monomorphism

3) $T - \lambda Id$ is not an isomorphism

4) $[T - \lambda Id]_B$ is not invertible $\forall B$

5) $[T - \lambda Id]_B$ is not invertible for some B.

6) $\det [T - \lambda Id]_B = 0$ for some basis B.

Proof: 1) $\Rightarrow$ 2)

λ is an eigenvalue $\iff \exists\, v \neq 0 : T(v) = \lambda v$

$\iff \exists\, v \neq 0 : (T - \lambda Id)(v) = 0$

$\iff Ker(T - \lambda Id) \neq 0$

$\iff T - \lambda Id$ is not a monomorphism.

2) $\Rightarrow$ 3) $(T : V \to W,\ \dim V = \dim W : T$ mono $\iff T$ epi $\iff T$ iso$)$.

Lemma: If $T \in Hom_F(V, V)$, B, B' are basis of V, then $\det [T]_B = \det [T]_{B'}$

Proof: $[T]_{B'} = [B]_{B'} [T]_B [B']_B = [B']_B^{-1} [T]_B [B']_B$

$\det [T]_{B'} = \det \left([B']_B^{-1} [T]_B [B']_B \right) = \det [B']_B^{-1} \cdot \det [T]_B \cdot \det [B']_B = (*)$

If $C \cdot C^{-1} = Id$, $C \in M_n(F) \Rightarrow 1 = \det Id = \det(C \cdot C^{-1}) = \det C \cdot \det C^{-1}$

$\Rightarrow \det C^{-1} = (\det C)^{-1} = \frac{1}{\det C} \in F$.

$(*) = \left(\det [B']_B\right)^{-1} \cdot \det [T]_B \cdot \det [B']_B = \det [T]_B \cdot \left(\det [B']_B\right)^{-1} \cdot \det [B']_B$

$= \det [T]_B$.

$\blacksquare$

Remark: The previous Lemma is true for any linear map.

In particular, take $T - \lambda Id : V \to V$. B, B' basis for V.

then $\det [T - \lambda Id]_B = \det [T - \lambda Id]_{B'}$

Def: Let $T \in Hom_F(V, V)$. The characteristic polynomial associated to T is given by $P_T(x) = \det [x Id - T]_B$

Theorem: λ is an eigenvalue of $T \iff \lambda$ is a root of $P_T(x)$.

Proof: λ root of $P_T(x) \iff 0 = P_T(\lambda) = \det[\lambda Id - T]_B$

$$= (-1)^n \det[T - \lambda Id]_B \underset{(1)\iff(6)}{\iff} \lambda \text{ is an eigenvalue.}$$

∎

Example:

1) $T(x,y) = (x+y, x+y)$

$[T]_B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $P_T(x) = \left| [xId - T]_B \right| = \left| [xId]_B - [T]_B \right| = \left| x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right|$

Defined?

$$= \begin{vmatrix} x-1 & -1 \\ -1 & x-1 \end{vmatrix} = (x-1)^2 - 1 = x(x-2)$$

Eigenvalue for T: $\lambda = 0$, $\lambda = 2$.

$V_0 = \{v : T(v) = 0 \cdot v\} = \operatorname{Ker} T = \{(x,y) \in F^2 . \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\} = \{(x,y) : x+y=0\}$

$$= \{(x,-x), x \in F\} = \langle (1,-1) \rangle.$$

$V_2 = \{v : T(v) = 2v\} = \{v : (T - 2Id)(v) = 0\} = \operatorname{Ker}(T - 2Id)$

$$= \left\{(x,y) \in F^2 : \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$$

$$= \left\{(x,y) \in F^2 : \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \{(x,y) : x - y = 0\}$$

$$= \{(x,x), x \in F\} = \langle (1,1) \rangle.$$

2) $[T]_B = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, $P_T(x) = \begin{vmatrix} x-3 & -1 & 1 \\ -2 & x-2 & 1 \\ -2 & -2 & x \end{vmatrix} = (x-1)(x-2)^2$

Eigenvalues : 1 , 2 .

$$V_1 = \ker(T - Id) = \left\{ (x,y,z) : \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \{ (x, 0, 2x) . x \in F \}$$

$$= \langle (1, 0, 2) \rangle .$$

$$V_2 = \{ v : T(v) = 2v \} = \ker(T - 2Id) = \left\{ (x,y,z) : \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} .$$

$$= \{ (x, x, 2x) , x \in F \} = \langle (1, 1, 2) \rangle .$$

$T: V \longrightarrow V$, FIND B : $[T]_B$ IS "EASY", FOR INSTANCE, $[T]_B$ = DIAGONAL = $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

λ EIGENVALUE OF $T \iff \exists v$ EIGENVECTOR FOR λ : $v \neq 0$ AND $T(v) = \lambda . v$.

HOW CAN WE FIND EIGENVALUES?

$P_T(x) = \det([xId - T]_B)$, $\forall B$ BASIS : λ EIGENVALUE OF $T \iff P_T(\lambda) = 0$.

EIGENVECTORS?

$V_\lambda = \{0\} \cup \{$EIGENVECTORS OF $\lambda\} = \{v \in V : T(v) = \lambda v\}$ = SUBSPACE = $\{v \in V : (T - \lambda Id)(v) = 0\} = Ker(T - \lambda Id)$

λ EIGENVALUE $\Longrightarrow V_\lambda \neq \{0\} \Longrightarrow \dim V_\lambda \geq 1$.

Def: A linear map $T: V \to V$ is diagonalizable if $\exists B$ basis of V $[T]_B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Remarks: 1) : T is diagonalizable $\iff \exists \lambda_1, \lambda_2, \dots, \lambda_n \in F$, $B = \{v_1, v_2, \dots, v_n\}$.
$T(v_i) = \lambda_i v_i$ $\forall i = 1, \dots, n$ $\iff \exists B$ a basis of eigenvectors.

2) : If T is diagonalizable $\Rightarrow P_T(x) = |[xId - T]_B| = |xId_n - [T]_B|$

$$= \begin{vmatrix} x-\lambda_1 & & & 0 \\ & x-\lambda_2 & & \\ & & \ddots & \\ 0 & & & x-\lambda_n \end{vmatrix} = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \in F[x]$$

$\Rightarrow P_T(x)$ has all its roots in F.

EXAMPLES:

1) $[T]_\mathcal{E} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $P_T(x) = \begin{vmatrix} x-1 & -1 \\ 0 & x-1 \end{vmatrix} = (x-1)^2 \longrightarrow \lambda = 1$ IS THE UNIQUE EIGENVALUE.

$V_1 = \{(x,y) : T(x,y) = 1\cdot(x,y)\} = \{(x,y) : \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}\} = \{(x,y) : \begin{cases} x+y = x \\ y = y \end{cases}\} = \{(x,0), x \in F\} = \langle (1,0) \rangle$.

EIGENVECTORS = $\{(x,0), x \neq 0\}$, $\nexists B$ BASIS OF EIGENVECTORS

$\Longrightarrow T$ IS NOT DIAGONALIZABLE

2) $T(x,y) = (x+y, x+y)$, $[T]_\mathcal{E} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $P_T(x) = x(x-2)$

$V_0 = \langle (1,-1) \rangle$, $V_2 = \langle (1,1) \rangle \longrightarrow B = \{(1,-1),(1,1)\}$ BASIS AND $[T]_B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

3) $[T]_\mathcal{E} = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, $P_T(x) = (x-1)(x-2)^2$, $V_1 = \langle (1,0,2) \rangle$, $V_2 = \langle (1,1,2) \rangle$

T IS NOT DIAGONALIZABLE SINCE $\nexists B$ BASIS OF EIGENVECTORS.

$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} \quad \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$

4) $T(x,y) = (-y, x)$, $[T]_\mathcal{E} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $P_T(x) = \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2 + 1$

$T: \mathbb{R}^2 \to \mathbb{R}^2 \Longrightarrow T$ IS NOT DIAGONALIZABLE SINCE IT HAS NO EIGENVALUES

$T: \mathbb{C}^2 \to \mathbb{C}^2 \Longrightarrow i, -i$ ARE EIGENVALUES

Remark: eigenvectors associated to different eigenvalues are L.I.

Proof: $v_1, v_2, \dots, v_m$: $T(v_i) = \lambda_i v_i$, $v_i \neq 0$, $\lambda_i \neq \lambda_j$ $\forall i \neq j$

By induction on m:

$m=1$, $a_1 v_1 = 0$, $v_1 \neq 0 \Rightarrow a_1 = 0$

$m=2$, $T(v_1) = \lambda_1 v_1$, $T(v_2) = \lambda_2 v_2$. $\lambda_1 \neq \lambda_2$

$$a_1 v_1 + a_2 v_2 = 0 \Rightarrow 0 = T(0) = T(a_1 v_1 + a_2 v_2) = \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2$$

$$= \lambda_1 \cdot 0 = \lambda_1 a_1 v_1 + \lambda_1 a_2 v_2$$

$$\Rightarrow \lambda_1 a_1 v_1 + \lambda_1 a_2 v_2 = \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2$$

$$\Rightarrow \underbrace{(\lambda_2 - \lambda_1)}_{\neq 0} a_2 \underbrace{v_2}_{\neq 0} = 0 \Rightarrow a_2 = 0 \Rightarrow a_1 v_1 = 0$$

Exercise for step m.

∎

Lemma: If $T \in Hom_F(V, V)$, λ eigenvalue, $v \neq 0$, $T(v) = \lambda v$, $P(x) = a_n x^n + \dots + a_0 \in F[x]$.

$\Rightarrow P(\lambda)$ is an eigenvalue of the linear map $P(T) = a_n T^n + \dots + a_1 T + a_0 Id_n : V \to V$ with eigenvalue v.

Proof: By induction, we can prove $T^k(v) = \lambda^k \cdot v$

- $k=1$, $T(v) = \lambda v$
- $k=2$, $T^2(v) = T(T(v)) = T(\lambda \cdot v) = \lambda \cdot T(v) = \lambda^2 \cdot v$.

Assume $T^{k-1}(v) = \lambda^{k-1} \cdot v \Rightarrow T^k(v) = T^{k-1}(T(v)) = T^{k-1}(\lambda \cdot v) = \lambda \cdot T^{k-1}(v) = \lambda^k \cdot v$.

$$(a_n T^n + \dots + a_1 T + a_0 Id)(v) = a_n \cdot T^n(v) + \dots + a_1 T(v) + a_0 \cdot v$$

$$= a_n \lambda^n \cdot v + \dots + a_1 \lambda \cdot v + a_0 \cdot v = P(\lambda) \cdot v.$$

∎

Theorem: Let $T \in Hom_F(V, V)$. let $\lambda_1, \cdots, \lambda_m$ be eigenvalues of T, $\lambda_i \neq \lambda_j$. $\forall i \neq j$

Then: 1): $W = V_{\lambda_1} + V_{\lambda_2} + \cdots + V_{\lambda_m} = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}$.

2): If B_i is a basis of $V_{\lambda_i} \Rightarrow B_1 \cup B_2 \cup \cdots \cup B_m$ is a basis of W.

3): $\dim W = \dim_{\lambda_1} + \cdots + \dim_{\lambda_m}$.

Proof: We know $S_1 + \cdots + S_m = S_1 \oplus \cdots \oplus S_m \Leftrightarrow$ any $v \in S_1 + \cdots + S_m$ can be written in a unique way. As $v = v_1 + \cdots + v_m$, $v_i \in S_i \Longleftrightarrow$

$0 = v_1 + v_2 + \cdots + v_m$, $v_i \in S_i \Rightarrow v_1 = v_2 = \cdots = v_m = 0$.

Let $v_1 + v_2 + \cdots + v_m = 0$, $v_i \in V_{\lambda_i}$, that is, $T(v_i) = \lambda_i v_i$. We have to prove $v_i = 0 \ \forall i$

Set $P_i(x) = \frac{(x-\lambda_1)}{(\lambda_i - \lambda_1)} \cdot \frac{(x-\lambda_2)}{(\lambda_i - \lambda_2)} \cdots \frac{(x-\lambda_{i-1})}{(\lambda_i - \lambda_{i-1})} \ (x - \lambda_{i+1}) \frac{(x-\lambda_{i+1})}{(\lambda_i - \lambda_{i+1})} \cdots \frac{(x-\lambda_m)}{(\lambda_i - \lambda_m)} \in F[x]$.

$P_i(\lambda_i) = 1 \quad P_i(\lambda_j) = 0 \quad j \neq i$

$0 = P_i(T)(v_1 + v_2 + \cdots + v_m) = \underbrace{P_i(\lambda_1)}_{=0} v_1 + \underbrace{P_i(\lambda_2)}_{=0} v_2 + \cdots + P_i(\lambda_i) v_i + \cdots + \underbrace{P_i(\lambda_m)}_{=0} v_m$.

$= v_i \quad \forall i$

▨

Theorem: Let $T \in Hom_F(V, V)$. The following are equivalent.

1) T is diagonalizable

2) $P_T(x) = (x-\lambda_1)^{C_1}(x-\lambda_2)^{C_2} \cdots (x-\lambda_m)^{C_m}$, $\lambda_i \neq \lambda_j \ \forall i \neq j$ ($n = \dim V = \deg P_T(x) = C_1 + C_2 + \cdots + C_m$)
and $\dim V_{\lambda_i} = C_i \quad \forall i = 1, \cdots, m$

3) $\dim V = \dim V_{\lambda_1} + \dim V_{\lambda_2} + \cdots + \dim V_{\lambda_m}$

4) $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}$

Proof: 1) $\Rightarrow$ 2) T diag. $\Rightarrow \exists B$ a basis : $[T]_B = \begin{bmatrix} \lambda_1 & & & & \\ & \ddots \lambda_1 & & & \\ & & \ddots \lambda_m & & \\ & & & \ddots & \\ & & & & \lambda_m \end{bmatrix}$ (c_1 copies of λ_1, …, c_m copies of λ_m)

$\Rightarrow P_T(x) = (x-\lambda_1)^{c_1} \cdots (x-\lambda_m)^{c_m}$

Moreover, $\dim V_{\lambda_i} \geq c_i$ since the basis B contains c_i L.I. associated to λ_i

We know : $V \supseteq V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m}$

$\Rightarrow n = \dim V \geq \dim V_{\lambda_1} + \dim V_{\lambda_2} + \cdots + \dim V_{\lambda_m} \geq c_1 + c_2 + \cdots + c_m = n$

If $\dim V_{\lambda_i} > c_i \Rightarrow n \geq \dim V_{\lambda_1} + \cdots + \dim V_{\lambda_m} > c_1 + c_2 + \cdots + c_m = n$

$\Rightarrow \dim V_{\lambda_i} = c_i$

2) $\Rightarrow$ 3) $n = \deg P_T(x) = c_1 + \cdots + c_m$.

$\Rightarrow \dim V = n = c_1 + \cdots + c_m = \dim V_{\lambda_1} + \cdots + \dim V_{\lambda_m}$.

3) $\Rightarrow$ 4) We know : $V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_m} \subseteq V$

and $\dim (V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_m}) = \dim V_{\lambda_1} + \cdots + \dim V_{\lambda_m} = \dim V = n$.

$\Rightarrow V_{\lambda_1} \oplus \cdots + V_{\lambda_m} = V$.

4) $\Rightarrow$ 1) If $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_m} \Rightarrow \exists B$ a basis of B such that

$B = B_1 \cup B_2 \cup \cdots \cup B_m$. B_i basis of V_{λ_i}

B_i = basis of eigenvectors associated to λ_i

$\Rightarrow B$ is a basis of eigenvectors $\Rightarrow T$ is diagonalize. $\blacksquare$

Corollary: If $T \in Hom_F(V,V)$, $P_T(x) = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n) \in F[x]$, $\lambda_i \neq \lambda_j$ $\forall i \neq j \Rightarrow T$ is diagonalizable.

Proof: We know that $\dim V_{\lambda_i} \geq 1$

Moreover, $V \supseteq V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n} \Rightarrow n = \dim V \geq \dim V_{\lambda_1} + \cdots + \dim V_{\lambda_n}$

$\geq 1 + 1 + \cdots + 1 = n \Rightarrow \dim V_{\lambda_i} = 1$ $\forall i$

▨

Def: λ eigenvalue of T

Algebraic multiplicity of λ = multiplicity of the root λ in $P_T(x)$.

Geometric multiplicity of λ = $\dim V_\lambda$.

Theorem (*): T is diagonalizable $\Longleftrightarrow$ $P_T(x)$ has all its roots in F and Geom. mult. (λ) = ALG. mult. (λ) $\forall \lambda : P_T(\lambda) = 0$

Remark: 1): Geom. mult. $(\lambda) \geq 1$ and ALG. mult. $(\lambda) \geq 1$ $\forall \lambda$: root of $P_T(x)$.

2): For any eigenvalue λ, Geo. mult. $(\lambda) \leq$ ALG. mult. (λ).

6.9 Lecture. (60 – 61)

Def: Given $A, B \in M_n(F)$, we say that A <u>is similar to</u> B if $\exists\, C \in M_n(F)$, C invertible : $A = C^{-1} \cdot B \cdot C$, notation $A \sim B$.

Proposition: "similar" is an equivalent relation $A \sim B \Leftrightarrow \exists\, C : A = C^{-1} \cdot B \cdot C$.

Proof: $A \sim A \Leftrightarrow \exists\, Id : A = Id^{-1} \cdot A \cdot Id$.

; $A \sim B \Leftrightarrow \exists\, C : A = C^{-1} \cdot B \cdot C \Leftrightarrow \exists\, C^{-1} : B = (C^{-1})^{-1} A \cdot (C^{-1}) \Leftrightarrow B \sim A$.

$A_1 \sim A_2, A_2 \sim A_3 \Leftrightarrow \exists\, C_1, C_2 : A_1 = C_1^{-1} A_2 \cdot C_1$, $A_2 = C_2^{-1} \cdot A_3 \cdot C_2$

$\Leftrightarrow A_1 = (C_2 C_1)^{-1} A_3 (C_2 C_1) \Rightarrow A_1 \sim A_3$

∎

Theorem: $A_1 \sim A_2 \Leftrightarrow A_1$ and A_2 are associated to the same linear map $T: F^n \to F^n \Leftrightarrow \exists\, T: F^n \to F^n : [T]_{B_1} = A_1$ and $[T]_{B_2} = A_2$, for B_1, B_2 basis of F^n.

Proof: $\Leftarrow$) Assume $[T]_{B_1} = A_1$, $[T]_{B_2} = A_2$. then $A_1 = [T]_{B_1} = [B_2]_{B_1} [T]_{B_2} [B_1]_{B_2}$

$= ([B_1]_{B_2})^{-1} \cdot A_2 \cdot [B_1]_{B_2}$.

$\Rightarrow A_1 \sim A_2$.

$\Rightarrow$). Assume $A_1 \sim A_2$. Define $T: F^n \to F^n$ s.t. $A_2 = [T]_{B_2}$ for B_2 any basis of F^n

$$A_1 \sim A_2 \Rightarrow \exists\, C \text{ invertible} : A_1 = C^{-1} \cdot A_2 \cdot C = C^{-1} [T]_{B_2} \cdot C$$
$$= ([B_1]_{B_2})^{-1} \cdot [T]_{B_2} \cdot [B_2]_{B_1}.$$
$$= [T]_{B_1}$$

$\blacksquare$

Remark : A diagonalizable $\overset{\text{Def}}{\Longleftrightarrow} \exists\, C : D = C^{-1} \cdot A \cdot C$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

$\overset{\text{Def}}{\Longleftrightarrow} A \sim \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \overset{\text{Theo}}{\Longleftrightarrow} A = [T]_{B_1}, \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = [T]_{B_2}$

$\overset{\text{Def}}{\Longleftrightarrow}$ T is diagonalizable.

Proposition : $A_1 \sim A_2 \Longrightarrow P_{A_1}(x) = P_{A_2}(x)$, where $P_A(x) = \det[x Id - A]$.

Proof : $A_1 \sim A_2 \overset{\text{Theo}}{\Longrightarrow} \exists\, T : F^n \to F^n : A_1 = [T]_{B_1}$. $A_2 = [T]_{B_2}$.

$P_{A_i}(x) = \det(x Id - A_i) = \det([x Id - T]_{B_i}) = P_T(x)$. $i = 1, 2$.

$\Rightarrow P_{A_1}(x) = P_T(x) = P_{A_2}(x)$.

$\blacksquare$

Remark : $\not\Leftarrow$: counter example : $A_1 = Id \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $P_{A_1}(x) = \begin{vmatrix} x-1 & 0 \\ 0 & x-1 \end{vmatrix} = (x-1)^2$

$P_{A_2}(x) = \begin{vmatrix} x-1 & -1 \\ 0 & x-1 \end{vmatrix} = (x-1)^2$

$A_3 \sim A_1 \overset{\text{Def}}{\Longleftrightarrow} \exists\, C$: invertible : $A_3 = C^{-1} \cdot A_1 \cdot C = C^{-1} \cdot Id \cdot C = Id$

Hence $A_2 \nsim A_1$.

Theorem: If A_1, A_2 are diagonalizable, then $A_1 \sim A_2 \iff P_{A_1}(x) = P_{A_2}(x)$.

Proof: $\Rightarrow$) Previous proposition.

$\Leftarrow$) $P_{A_1}(x) = P_{A_2}(x)$, $C_1^{-1} \cdot A_1 C_1 = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, $C_2^{-1} \cdot A_2 \cdot C_2 = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}$

$$P_{C_i^{-1}A_iC_i}(x) = \det(x Id - C_i^{-1}A_iC_i) = \det(x \cdot C_i^{-1} \cdot Id \cdot C_i - C_i^{-1} \cdot A_iC_i)$$

$$= \det(C_i^{-1}(x \cdot Id - A_i) \cdot C_i)$$

$$= \det C_i^{-1} \cdot \det(x Id - A_i) \cdot \det C_i = \det C_i^{-1} \cdot \det C_i \cdot P_{A_i}(x)$$

$$= \det(C_i^{-1} \cdot C_i) \cdot P_{A_i}(x) = P_{A_i}(x).$$

$$\begin{bmatrix} x-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & x-\lambda_n \end{bmatrix} = P_{A_1}(x) = P_{A_2}(x) = \begin{bmatrix} x-\mu_1 & & 0 \\ & \ddots & \\ 0 & & x-\mu_2 \end{bmatrix}$$

$$\Rightarrow (x-\lambda_1) \cdots (x-\lambda_n) = (x-\mu_1) \cdots (x-\mu_n)$$

$$\Rightarrow \{\lambda_1, \cdots, \lambda_n\} = \{\mu_1, \cdots, \mu_n\}.$$

$$A_1 \sim \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \sim \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{pmatrix} \sim A_2.$$

$\blacksquare$

Theorem: Let $T: V \to V$ s.t. $P_T(x)$ has all its roots in F: $P_T(x) = (x-\lambda_1) \cdots (x-\lambda_n)$, $\lambda_i \in F$

then $\exists B$ a basis of V s.t. $[T]_B$ is upper triangular, that is, $[T]_B = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

Proof: By induction on $n = \dim V$.

$n=1$, $P_T(x) = (x-\lambda_1) \Rightarrow [T]_B = [\lambda_1] \Rightarrow T = \lambda_1 Id$.

Assume true for $n-1$

λ_1 is a root of $P_T(x) \Rightarrow \lambda_1$ is eigenvalue $\Rightarrow \exists v_1 \neq 0,\ T(v_1) = \lambda_1 v_1$.

Extended $\{v_1\}$ to a basis $B = \{v_1, v_2, \cdots v_n\} \Rightarrow [T]_B = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & & \\ \vdots & \vdots & & & \\ 0 & a_{n2} & \cdots & & \end{pmatrix}$

Let $W = \text{span}(v_2, v_3, \cdots, v_n) \Rightarrow \dim W = n-1$.

$T' : W \to W : [T']_{B'} = \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $B' = \{v_2, \cdots, v_n\}$.

$$T(v_2) = a_{12}v_1 + \underbrace{a_{22}v_2 + a_{32}v_3 + \cdots + a_{n2}v_n}_{T'(v_2)} = a_{12}v_1 + T'(v_2).$$

$\vdots$

$$T(v_n) = a_{1n}v_1 + T'(v_n).$$

$$\Rightarrow [T]_B = \left(\begin{array}{c|c} \lambda_1 & a_{12} \cdots a_{1n} \\ \hline 0 & \\ \vdots & [T']_{B'} \\ 0 & \end{array}\right) \Rightarrow P_T(x) = \left|\begin{array}{c|c} x-\lambda_1 & -a_{12} \cdots -a_{1n} \\ \hline 0 & \\ \vdots & [xId - T']_{B'} \\ 0 & \end{array}\right| = (x-\lambda_1)\cdot \det([xId - T']_{B'})$$
$$= (x-\lambda_1)\cdot P_{T'}(x).$$

$\Rightarrow P_{T'}(x) = (x-\lambda_2)\cdots(x-\lambda_n)$. By I.H., $\exists B'' = \{w_2, \cdots, w_n\}$, $[T']_{B''} = \begin{pmatrix} \lambda_2 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

PRODUCT OF VECTOR SPACES

DEF: IF V AND W ARE TWO F- VECTOR SPACES, THE PRODUCT

$$V \times W = \{ (v, w),\ v \in V,\ w \in W \}$$ IS THE F- VECTOR SPACE

WITH ADDITION: $(v, w) + (v', w') = (v + v', w + w')$

AND PRODUCT: $\lambda (v, w) = (\lambda v, \lambda w)$.

EXERCISE: CHECK THAT $V \times W$ IS A VECTOR SPACE.

WE CAN EXTEND THIS DEFINITION FOR ANY ARBITRARY FAMILY OF VECTOR SPACES.

$$V_1 \times V_2 \times \dots \times V_n = \{ (v_1, v_2, \dots, v_n),\ v_i \in V_i \}$$

$$\prod_{i \in \mathbb{N}} V_i = V_1 \times V_2 \times \dots = \{ f : \mathbb{N} \to \bigcup V_i \ : \ f(i) = v_i \in V_i \} = \{ (v_1, v_2, \dots),\ v_i \in V_i \}$$

$$\{ V_i,\ i \in I \},\quad \prod_{i \in I} V_i = \{ f : I \to \bigcup V_i \ : \ f(i) \in V_i \}$$

EXAMPLE: $F^n = \underbrace{F \times F \times \dots \times F}_{n\text{-TIMES}}$

THEO: LET $T : V \to V$ SUCH THAT $P_T(x)$ HAS ALL ITS ROOTS IN F: $P_T(x) = (x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_n)$, $\lambda_i \in F$.
THEN $\exists B$ A BASIS OF V SUCH THAT $[T]_B$ IS UPPER TRIANGULAR, THAT IS $[T]_B = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

PROOF: BY INDUCTION ON $n = \dim V$

$n = 1$, $P_T(x) = (x - \lambda_1) \Rightarrow [T]_B = [\lambda_1] \Rightarrow T = \lambda_1 Id$

ASSUME TRUE FOR $n-1$.

λ_1 IS A ROOT OF $P_T(x) \Rightarrow \lambda_1$ IS AN EIGENVALUE $\Rightarrow \exists v_1 \neq 0$, $T(v_1) = \lambda_1 v_1$

WHAT IS THE CONNECTION BETWEEN PRODUCT AND DIRECT SUM?

THEO: IF $S_1, S_2, \dots, S_n$ ARE SUBSPACES OF V, THEN

$$S_1 \times S_2 \times \dots \times S_n \simeq S_1 \oplus S_2 \oplus \dots \oplus S_n.$$

PROOF: $S_1 \oplus S_2 \oplus \dots \oplus S_n \xrightarrow{\varphi} S_1 \times \dots \times S_n$

$v = v_1 + v_2 + \dots + v_n \longrightarrow (v_1, v_2, \dots, v_n)$

(IN A UNIQUE WAY)

LINEAR MAP

EPIMORPHISM

MONOMORPHISM: $\operatorname{Ker} \varphi = \{ v : v_1 = v_2 = \dots = v_n = 0 \} = \{0\}$.

NOT TRUE FOR ARBITRARY FAMILIES! WE CAN ALWAYS DEFINE A MONOMORPHISM $\bigoplus_{i \in I} S_i \longleftrightarrow \prod_{i \in I} S_i$

$v = \sum v_i,\ v_i \in S_i \longrightarrow (v_i)_{i \in I}$

$S \oplus S \oplus S \oplus \dots \longrightarrow S \times S \times S \times \dots$

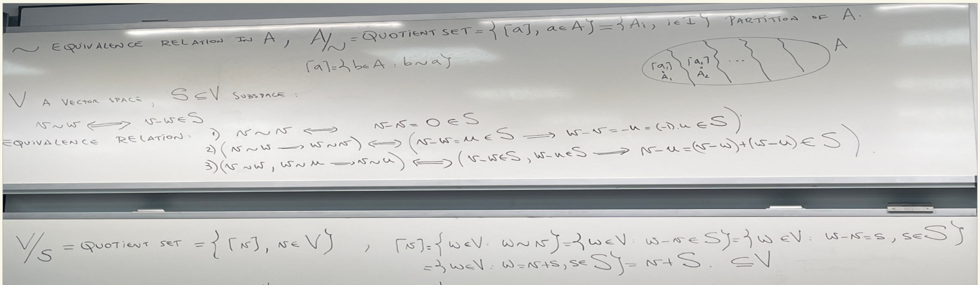
$\sum v_i,\ v_i \in S_i$ FINITE $\not\longrightarrow (v, v, v, \dots)\ v \neq 0$

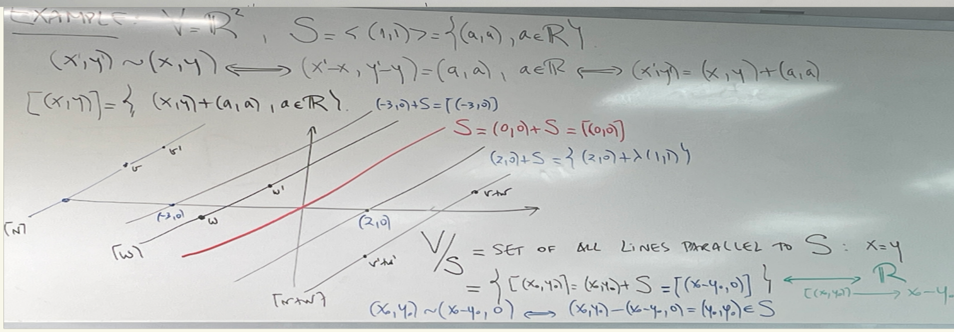
QUOTIENT

DUAL SPACE

SYSTEM OF LINEAR EQUATIONS $\longleftrightarrow$ SOLUTIONS

6.11. Lecture





Theorem: If S is a subspace of V, then V/S is a vector space.

$$[v] + [w] = [v+w] \qquad \lambda\cdot[v] = [\lambda\cdot v]$$

Proof: Well Defined: $v \sim v',\ w \sim w' \Rightarrow v - v' \in S,\ w - w' \in S$

$\Rightarrow v+w + (v'-w') \in S.$

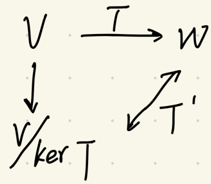
$[v] = [v'],\ [w] = [w'] \Rightarrow [v+w] = [v'+w'].$

$v \sim v' \Rightarrow v - v' \in S \Rightarrow \lambda v - \lambda v' = \lambda(v - v') \in S \Rightarrow \lambda v \sim \lambda v'$

∎

Theorem: Let $T: V \to W$ be a linear map. $\ker T \leq V$ subspace.

Then $\exists!\, T': V/\ker T \to W$ a monomorphism making the following diagram commtative:



Proof: $T'([v]) = T(v)$

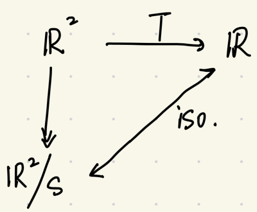
Well defined: $v \sim v' \Longrightarrow v - v' \in \ker T \Rightarrow T(v) = T(v')$.

$[v] = [v'] \Longrightarrow T'([v]) = T'([v'])$

T' linear map

$\blacksquare$

Corollary: $\tilde{T}: V/\ker T \to \operatorname{Im} T$ isomorphism. $\tilde{T}([v]) = T'([v]) = T(v)$.



$T(x, y) = x - y$, $\ker T = S$

Theo: If $\dim V < \infty$, then $\dim V/S = \dim V - \dim S$

Proof: $T: V \to V/S$ epimorphism: $\dim V = \dim \ker T + \dim \operatorname{Im} T$

$v \to [v]$ $\qquad = \dim S + \dim V/S$.

$\ker T = \{v : [v] = [0]\} = \{v : v - 0 \in S\} = S$.

Duality