Workshop

Week 8

December 6, 2024

1. Use Bolzano's theorem to prove that the following equations has at least one real solution.
   (a) $e^x + x + 2 = 0$
   (b) $x^3 + \ln(x) = -\sqrt{x}$
2. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.
3. Choose an interval $[a, b]$ such that the function $f : [a, b] \to \mathbb{R}$ will be injective. Give the inverse function in each case.
   (a) $f(x) = x^2$
   (b) $f(x) = \dfrac{1}{x}$
4. Compute the inverse of the following functions
   (a) $f(x) = -2\ln\left(\dfrac{x+1}{x-1}\right)$
   (b) $f(x) = \begin{cases} \frac{1}{2}x \text{ if } x < 0 \\ 2x \text{ if } x \geq 0 \end{cases}$
   (c) $f(x) = \begin{cases} -x^2 \text{ if } x \geq 0 \\ 1 - x^3 \text{ if } x < 0 \end{cases}$
5. Using the definition of ln prove that
   (a) $\ln(ab) = \ln(a) + \ln(b)$
   (b) $\ln\left(\dfrac{a}{b}\right) = \ln(a) - \ln(b)$
   (c) $\ln(a^b) = b\ln(a)$