Tutorial

Week 4

November 5, 2024

1. Compute the following limits

   (a) $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

   (b) $\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^n$

   (c) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n$

   (d) $\lim_{n\to\infty}\frac{n!}{n^n}$

2. Prove using the definition of Cauchy sequence that if (x_n) and (y_n) are Cauchy sequences, then (x_n+y_n), $(x_n.y_n)$ are Cauchy sequences.

3. Prove that if x_n is a convergent sequence (or a Cauchy sequence), then the sequence $|x_{n+1}-x_n|$ converges to 0.

   The converse is not true: find an example of a non convergent sequence with this property.

4. Let (x_n) be a sequence, prove that if $0\le r<1$, $C>0$ and $|x_{n+1}-x_n|<Cr^n$, then (x_n) is a Cauchy sequence.

5. Show that the sequence (x_n) defined below is a Cauchy sequence.

   $x_1=1$ and $x_{n+1}=1+\frac{1}{x_n}$ for all $n\ge 1$.

6. Prove that if $\sum a_n$ converges then $\lim_{n\to 0} a_n=0$. Give a counterexample explaining why the converse is not true.

7. Prove the Algebraic Limit Theorem for Series: Let $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then

   (a) $\sum_{n=1}^{\infty} ca_n = cA$ for all $c\in\mathbb{R}$.

(b) $\sum_{n=1}^{\infty}(a_n + b_n) = A + B$

8. Analize the following geometric series:

   (a) $\sum_{n=3}^{\infty} \frac{1}{3^n}$

   (b) $\sum_{n=2}^{\infty} \frac{(-5)^n}{8^{n+2}}$

   (c) $\sum_{n=1}^{\infty} \frac{10^n + 5^{2n}}{6^{2n}}$

9. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.