Tutorial

Week 1

October 17, 2024

1. For $a \neq 0$, prove that:
   (a) $(a^{-1})^{-1} = a^{-1}$
   (b) $(ab)^{-1} = a^{-1}b^{-1}$
2. Using axiom of order ($\leq$), prove the following properties for $<$.
   (a) If $a < b$ and $b < c$ then $a < c$.
   (b) If $a < b$ and then $a + c < b + c$.
   (c) If $a < b$ and $c > 0$ then $ac < bc$.
3. Show that $c > 0$ if and only if $-c < 0$ for all $c \in \mathbb{R}$.
4. Sign rules: Show the the following sentences.
   (a) If $a > 0$ and $b > 0$ then $ab > 0$.
   (b) If $a > 0$ and $b < 0$ then $ab < 0$.
   (c) If $a < 0$ and $b > 0$ then $ab < 0$.
   (d) If $a < 0$ and $b < 0$ then $ab > 0$.
5. Prove that if $a < b$ and $c < 0$ then $ac > bc$.
6. Prove that $1 > 0$.
7. Show that:
   (a) $a > 0$ if and only if $a^{-1} > 0$.
   (b) $a < 0$ if and only if $a^{-1} < 0$.
   (c) If $0 < a < b$ then $a^{-1} > b^{-1}$.
8. Prove that:

(a) $ab > 0$ if and only if $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$.

(b) $ab < 0$ if and only if $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

9. Find the solution set of the following inequalities:

   (a) $4 - x < 3 - 3x$

   (b) $x^2 > 9$

   (c) $\frac{x-1}{x+1} > 0$

   (d) $\frac{1}{x} + \frac{1}{1-x} > 0$

10. Show that:

    (a) $|a.b| = |a|.|b|$

    (b) $|a| - |b| \leq |a - b|$

    (c) $||a| - |b|| \leq |a - b|$

11. Let $a, b \in \mathbb{R}$ and $b \geq 0$, then $-b \leq a \leq b$ if and only if $|a| \leq b$. In particular, $-|a| \leq a \leq |a|$.

12. Show that the supremum and infimum of a set are unique.

13. (a) Prove that if $M \in A$ is an upper bound for A, then $M = sup\, A$.

    (b) (Workshop) Prove that if $m \in A$ is a lower bound for A, then $m = inf\, A$.

14. Assume $A \subseteq \mathbb{R}$ and $A \neq \emptyset$. We denote $-A := \{-a : a \in A\}$.

    (a) If A has a lower bound then $-A$ has an upper bound and $\inf(A) = -sup(-A)$.

    (b) (Workshop) If A has an upper bound then $-A$ has a lower bound and $inf(-A) = -sup(A)$.

15. Show that if $sup\, A < sup\, B$ then exists $b \in B$ that is an upper bound of A.

16. (a) Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = sup\, A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

    (b) (Homework) Assume $i \in \mathbb{R}$ is an lower bound for a set $A \subseteq \mathbb{R}$. Then $i = inf\, A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $a < \epsilon + i$.

17. For each of the following sets compute the supremum and infimum. Justify.

    (a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(b) Let α and β be real numbers such that $\alpha < \beta$

$$\{x \in \mathbb{Q} : \alpha < x < \beta\}$$

(c) $\{x \in \mathbb{Q} : -\frac{3}{4} \leq x \leq 0\}$

(d) $\{x \in \mathbb{Q} : 0 < x^2 < 2\}$

(e) $\left\{\dfrac{n}{2n+1} : n \in \mathbb{N}\right\}$