Tutorial

Week 2

October 23, 2024

1. (a) Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = sup\, A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

   (b) Assume $i \in \mathbb{R}$ is an lower bound for a set $A \subseteq \mathbb{R}$. Then $i = inf\, A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $a < \epsilon + i$.

2. Show that if $sup\, A < sup\, B$ then exists $b \in B$ that is an upper bound of A.

3. For each of the following sets compute the supremum and infimum. Justify.

   (a) $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$

   (b) $\left\{ \frac{n}{2n+1} : n \in \mathbb{N} \right\}$

   (c) $\{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$

   (d) Let α and β be real numbers such that $\alpha < \beta$

   $$\{x \in \mathbb{Q} : \alpha < x < \beta\}$$

   (e) $\{x \in \mathbb{Q} : -\frac{3}{4} \leq x \leq 0\}$

4. Recall that a set A is dense in $\mathbb{R}$ if an element of A can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

   (a) The set of all rational numbers p/q with q a power of 2.

   (b) The set of all rational numbers p/q with $q \leq 10$.

   (c) The set of all rational numbers p/q with $10|p| \geq q$.