Tutorial

Week 3

October 31, 2024

1. **Algebra of limits:**

   (a) Let $\lim_{n\to\infty} a_n = a$, and $\lim_{n\to\infty} b_n = b$. Then,

      i. $\lim_{n\to\infty} ca_n = a$, for all $c \in \mathbb{R}$;

      ii. $\lim_{n\to\infty} a_n + b_n = a + b$

      iii. $\lim_{n\to\infty} a_n b_n = ab$;

      iv. If $a \neq 0$, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$

      v. If $b \neq 0$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

      vi. If $a_n \leq b_n$ for all n, then $a \leq b$,

   (b) If $\lim_{n\to\infty} a_n = 0$,and $|b_n| < C$ for all n then $\lim_{n\to\infty} a_n b_n = 0$.

2. **We say that $\lim_{n\to\infty} a_n = \infty$ if for any positive real number M there is a natural number N such that for every natural number $n \geq N$, we have $a_n > M$.**

   We say that $\lim_{n\to\infty} a_n = -\infty$ if for any negative real number K there is a natural number N such that for every natural number $n \geq N$, we have $a_n < K$.

   Prove that:

   (a) If $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} b_n = b \neq 0$ then $\lim_{n\to\infty} a_n b_n = \begin{cases} \infty, \text{ if } b > 0 \\ -\infty, \text{ if } b < 0 \end{cases}$.

   (b) if $\lim_{n\to\infty} a_n = -\infty$ and $\lim_{n\to\infty} b_n = b \neq 0$ then $\lim_{n\to\infty} a_n b_n = \begin{cases} -\infty, \text{ if } b > 0 \\ \infty, \text{ if } b < 0 \end{cases}$

   Proof of (a): Suppose $b > 0$. We will use the following claim.

   Claim 1: If $\lim_{n\to\infty} b_n = b$ and $b > 0$, then exists δ and $N \in \mathbb{N}$ such that $b_n > \delta$ for all $n \geq$ N.

   Proof Claim 1: Exercise

   Since $\lim_{n\to\infty} b_n = b$ and $b > 0$, then there exist $0 < \delta$ and $N_1 \in \mathbb{N}$ such that $b_n > \delta$ for all $n \geq N_1$.

On the other hand, for a $M > 0$ (*), since $\lim_{n\to\infty} a_n = \infty$, exists $N_2 \in \mathbb{N}$ such that for every natural number $n \geq N_2$, we have $a_n > M$.

The following sentence is an useful tool, it is easy to prove

Exercise: If $0 < x < y$ and $0 < z < w$, then $0 < xz < yw$.

Let's continue with the proof. So we have now:

$$b_n > \delta > 0, \text{ and } a_n > M > 0$$

Then $a_n b_n > \delta.M > 0$ for $N = max\{N_1, N_2\}$. Now change M in (*) by M/δ, then $\lim_{n\to\infty} a_n b_n = \infty$.

Let's see the case $b < 0$. We will use the following claim:

Claim 2: Let $\lim_{n\to\infty} b_n = b$ and $b < 0$, then exists $\delta < 0$ and $N \in \mathbb{N}$ such that $b_n < \delta$ for all $n \geq$ N.

Proof of Claim 2: Exercise.

Since $\lim_{n\to\infty} b_n = b$ and $b < 0$, then there exist $\delta < 0$ and $N_1 \in \mathbb{N}$ such that $b_n < \delta$ for all $n \geq N_1$.

Suppose $K < 0$ (**), then $-K > 0$. Since $\lim_{n\to\infty} a_n = \infty$, exists $N_2 \in \mathbb{N}$ such that for every natural number $n \geq N_2$, we have $a_n > -K$.

Then $a_n > -K > 0$ and $b_n < \delta < 0$ for $N = max\{N_1, N_2\}$. Thus

$$\begin{aligned} a_n &> -K \\ a_n b_n &< -K b_n \\ a_n b_n < -K b_n &< \underbrace{-K\delta}_{<0} \end{aligned}$$

The last inequality holds for all $n \geq N$. You can change in (**) K by $K/|\delta|$, then $\lim_{n\to\infty} a_n b_n = -\infty$.

3. Find the limits of the following sequences:

   (a) $(a_n) = \dfrac{n-1}{n+1}$

   (b) $(b_n) = \dfrac{3n^2 + 2n + 2}{5n^2 + 4}$

   (c) $(c_n) = \dfrac{2n + 2}{5n^2 + 4}$

   (d) $(d_n) = \dfrac{3n^2 + 2n + 2}{5n + 4}$

   (e) $(e_n) = \dfrac{2^n + n^2}{2^n + 5}$

4. Prove that any subsequence of a convergent sequence is convergent.

5. (a) Let $0 < b < 1$, prove that $(a_n) = (b^n)$ is decreasing and bounded below, hence it has a limit b. Compute b.

   (b) Extend the result proved to the case $|b| < 1$; that is, show $\lim_{n\to\infty}(b^n) = 0$ if and only if $-1 < b < 1$.