

# Tutorial

Week 7

December 4, 2024

1. Use Bolzano's theorem to prove that the following equations has at least one real root.
  - (a)  $0 = x^3 + 5x^2 - 7x - 2$
  - (b)  $x^3 = 20 + \sqrt{x}$
  - (c)  $e^x + x + 2 = 0$
2. Prove that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial function of odd degree, then it necessarily has at least one real root. In particular, any real polynomial of degree 3 has (at least) one real root.
3. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous strictly increasing function (in particular it is injective). Use the mean value theorem to show that  $f$  considered as a map  $f : [a, b] \rightarrow [f(a), f(b)]$  is bijective.
4. Let  $g : [a, b] \rightarrow \mathbb{R}$  be a continuous bijective function.
  - (a) Show that if  $f(a) < f(b)$  then  $f(a) < f(x) < f(b)$  for all  $x \in (a, b)$ .
  - (b) Show that if  $f(a) > f(b)$  then  $f(a) > f(x) > f(b)$  for all  $x \in (a, b)$ .
5. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous and injective function.
  - (a) Show that if  $f(a) < f(b)$  then  $f$  is strictly increasing.
  - (b) Show that if  $f(a) > f(b)$  then  $f$  is strictly decreasing.
6. Prove that the exponential function is injective and its image is  $(0, \infty)$ .