Tutorial

Week 6

November 20, 2024

1. Use the ϵ-δ definition of limit to prove the following limits.

   (a) $\lim_{x \to a} x^2 = a^2$.

   (b) $\lim_{x \to 1} \frac{1}{x} = 1$.

   (c) $\lim_{x \to a} \sqrt{x} = \sqrt{a}$, $a > 0$.

2. Compute the following limits using factoring cases:

   (a) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

   (b) $\lim_{x \to 1} \frac{2x^2 - 4x + 2}{x - 1}$

   (c) $\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}$

3. Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

4. **Algebra of limits: Let f, g be real functions and x_0 is a limit point of $Domf$ and $Domg$. Suppose exist $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then**

   - $\lim_{x \to x_0} cf(x) = c \lim_{x \to x_0} f(x) = cL$
   - $\lim_{x \to x_0} (f + g)(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = L + M$
   - $\lim_{x \to x_0} (f.g)(x) = \lim_{x \to x_0} f(x). \lim_{x \to x_0} g(x) = L.M$
   - $\lim_{x \to x_0} \frac{1}{g}(x) = \frac{1}{M}$**, if** $M \neq 0$**.**

5. **Right and Left limits**

   - $\lim_{x \to a^+} f(x) = L_1$ **if for all** $\epsilon > 0$**, exists** $\delta > 0$ **such that for all** $x \in Domf \cap (a, a + \delta)$**, we have** $|f(x) - L_1| < \epsilon$**.**

- $\lim_{x\to a^-} f(x) = L_2$ **if for all** $\epsilon > 0$**, exists** $\delta > 0$ **such that for all** $x \in Domf \cap (a-\delta, a)$**, we have** $|f(x) - L_2| < \epsilon$**.**

  Prove that: $\lim_{x\to a} f(x)$ **exists if and only if** $\lim_{x\to a^+} f(x)$ **and** $\lim_{x\to a^-} f(x)$ **exist and are equal, moreover** $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$**.**

6. **Analyze the existence of** $\lim_{x\to 0} sgn(x)$**.**

7. **Let** g **be the following function**

$$g(x) = \begin{cases} 2 - x \textbf{ if } x < -1 \\ x + 2 \textbf{ if } -1 \leq x < 1 \\ \quad 4 \textbf{ if } x = 1 \\ 4 - x \textbf{ if } x > 1 \end{cases}$$

   Compute:

   (a) $\lim_{x\to -1^+} g(x)$

   (b) $\lim_{x\to -1^-} g(x)$

   (c) $\lim_{x\to 1^+} g(x)$

   (d) $\lim_{x\to 1^-} g(x)$

8. **Limits at infinity**

   - $\lim_{x\to\infty} f(x) = L$ **if for all** $\epsilon > 0$**, exists** $N > 0$ **such that whenever** $x > N$**, it follows that** $|f(x) - L| < \epsilon$**.**
   - $\lim_{x\to-\infty} f(x) = L$ **if for all** $\epsilon > 0$**, exists** $N < 0$ **such that whenever** $x < N$**, it follows that** $|f(x) - L| < \epsilon$**.**

   Prove that

   (a) $\lim_{x\to\infty} \frac{1}{x} = 0$

   (b) $\lim_{x\to-\infty} \frac{1}{x} = 0$

9. **Analyze** $\lim_{x\to\infty} \frac{P(x)}{Q(x)}$**, where** $P(x)$ **and** $Q(x)$ **are real polynomials.**