Homework 4

Week 4

November 9, 2024

1. Let (x_n) be a sequence such that $x_2 \neq x_1$ and satisfies the following contractive condition

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|,$$

   $\forall\, n \geq 1$ and for some $0 < C < 1$.

   (a) (10 points) Prove that $|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$ for all $n \geq 1$, by induction.

   (b) (15 points) Show the sequence (x_n) is a Cauchy sequence.

2. Consider the sequence (x_n) given by $x_1 = 2$ and $x_{n+1} = 2 + \dfrac{1}{x_n}$.

   (a) (10 points) Prove that (x_n) is bounded below.

   (b) (10 points) Show that (x_n) satisfies the contractive condition stated in the first exercise and hence it converges.

   (c) (5 points) Find $\lim\limits_{n \to \infty} x_n$.

3. Give an example for each of the following requests, or explain why the request is impossible to fulfill, referring the proper theorem(s).

   (a) (8 points) Two series $\sum x_n$ and $\sum y_n$ that are both divergent but $\sum x_n y_n$ converges.

   (b) (8 points) A convergent series $\sum x_n$ and a bounded sequence (y_n) such that $\sum x_n y_n$ diverges.

   (c) (9 points) Two sequences (x_n) and (y_n) where $\sum x_n$ and $\sum x_n y_n$ converge but $\sum y_n$ diverges.

4. Study the following series. Decide if they are convergent or not. If they are convergent, compute the value of the series.

(a) (15 points) $\displaystyle\sum_{n=1}^{\infty} -\frac{15}{(n+3)(n+4)}$

(b) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{9^{n-1}}$