Homework 3

Week 3

November 1, 2024

1. (a) (10 points) Prove that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.

   (b) (25 points) Show that the following sequence converges

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}} \cdots$$

   (c) (5 points) Find the limit. Hint: Note that $(x_{n+1})^2 = 2x_n$, where x_n denotes the n- term of the sequence.

2. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorem(s).

   (a) (8 points) Sequences (x_n) and (y_n) which both diverge but $(x_n + y_n)$ converges.

   (b) (8 points) Sequences (x_n) and (y_n) where (x_n) converges and (y_n) diverges but $(x_n + y_n)$ converges.

   (c) (8 points) A convergent sequence (x_n) with $x_n \neq 0$ for all n such that $\left(\frac{1}{x_n}\right)$ diverges.

   (d) (8 points) An unbounded sequence (x_n) and a convergent sequence (y_n) such that $(x_n - y_n)$ is bounded.

   (e) (8 points) Sequences (x_n) and (y_n) where (x_n), $(x_n y_n)$ converge but (y_n) diverges.

3. (20 points) Prove that for all real number $L \in (0, 1)$, exists a sequence of rational numbers (a_n) such that $a_n \in (0, 1)$ and $\lim_{n\to\infty} a_n = L$.