Homework 7

Week 8

December 7, 2024

1. Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function such that $f([0,1]) \subset \mathbb{Q}$. Show that f is a constant function.

2. Decide, for which $n \in \mathbb{N}$, if the function $p(x) = x^n$ is strictly increasing. Justify.

3. Prove that if f is strictly increasing, then so is its inverse.

4. Prove that the function $\sinh x = \dfrac{e^x - e^{-x}}{2}$ is bijective and compute the inverse.

5. Show that $f(x) = \dfrac{ax+b}{cx+d}$ is injective if and only if $ad - bc \neq 0$. Determine the Imf and find f^{-1}.

6. Let be $a > 0$, we define the exponential function $a^x := e^{\ln(a)x}$. Prove using the definition of exponential functions that

   (a) $a^{b+c} = a^b a^c$

   (b) $\left(a^b\right)^c = a^{bc}$

   (c) $a^n = a.a. \cdots a$ for all $n \in \mathbb{N}$

   (d) $a^{\frac{1}{n}} = \sqrt[n]{a}$ for all $n \in \mathbb{N}$

   (e) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ for all $n \in \mathbb{N}$, $m \in \mathbb{Z}$

   (f) a^x is strictly increasing for $a > 1$ and strictly decreasing for $0 < a < 1$.