Homework 6

Week 7

November 30, 2024

1. Prove using definition that
   (a) $\lim_{x\to 2} 3x^2 - 1 = 11$
   (b) $\lim_{x\to+\infty} \dfrac{1}{\sqrt{x}} = 0$
   (c) $\lim_{x\to 0^+} \dfrac{1}{\sin x} = +\infty$
2. (Contraction Mapping Theorem). Let$f$ be a function defined in all $\mathbb{R}$, and assume that there is a constant c such that $0 < c < 1$ and $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$.
   (a) Show that f is continuous on $\mathbb{R}$.
   (b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence $(y_1, f(y_1), f(f(y_1)), \cdots)$. In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence, let $y = \lim_{n\to\infty} y_n$.
   (c) Prove that y is a fixed point, that is $f(y) = y$. .
3. Consider the function $f(x) = \begin{cases} 5a^2x - 9 \text{ if } x > 3 \\ 9\sqrt{a} \text{ if } x = 3 \\ \dfrac{2x^2a}{3} \text{ if } x < 3 \end{cases}$
   Determine all values of a such that $\lim_{x\to 3} f(x)$ exists.
4. Compute the following limits
   (a) $\lim_{x\to -3} \dfrac{x^2 + 6x + 9}{\sin(x^2 - 9)}$
   (b) $\lim_{x\to 1} \dfrac{x^m - 1}{x^n - 1}$, $n, m \in \mathbb{N}$.
   (c) $\lim_{x\to+\infty} \dfrac{\sqrt{x + \sqrt{x + \sqrt{x}}}}{\sqrt{x + 1}}$

5. Decide if the following sentences are True or False

   (a) If $|f|$ is continuous in x_0 then f is continuous in x_0.

   (b) If $f(x)$ and $g(x) \neq 0$ for all x and $f^2(x) = g^2(x)$, then $f = g$ or $f = -g$.

   (c) There is a function $f(x)$ which is not continuous at 0 but $f(x) + \dfrac{1}{f(x)}$ is continuous at 0.