Student Feedback Report

Student Id	999027873
Exam Id	202411251041955
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Course Name	INFINITESIMAL CALCULUS 1 - Mid
Lecturer	Leandro CAGLIERO

Open question score	Original Exam Grade	**Final Exam Grade**
85.00	85.00	**85.00**

Summary

Question number	Actual points	Max points
1	20.00	20.00
2	19.00	20.00
3	16.00	20.00
4	13.00	20.00
5	17.00	20.00

✓ Correct answer ✓ Partial answer × Incorrect answer ⦸ Not answered

(57)
ID 999027873
Exam 202411251041955

Infinitesimal calculus I
104195

Midterm
November 25, 2024

Your ID Number: 999027873

Your Name: Yue Shi

Guidelines

- **Duration: 2 hours.** Use of calculators, personal dictionaries, electronic devices, reference materials, personal notes or any other extra material is not allowed.
- Show all your work. Explain your solutions, quote theorems you are using.
- Please, write clear and complete answers for each problem in the same page.
- No credit will be given for non-justified answers.

1. (20 points) Compute the supremum and the infimum of the following set. Justify.

$$\left\{ \frac{3n}{n+2} : n \in \mathbb{N} \right\}$$

Since $\frac{3n}{n+2} = \frac{3n+6-6}{n+2} = 3 - \frac{6}{n+2}$ and $n \in N$. then $1 \le 3 - \frac{6}{n+2} \le 3$

Let prove $\frac{3n}{n+2}$ is increasing

$a_{n+1} - a_n = \frac{3n+3}{n+3} - \frac{3n}{n+2} = \frac{\cancel{3n^2}+3n+6n+6 - \cancel{3n^2}-9n}{(n+3)(n+2)} = \frac{6}{(n+3)(n+2)} > 0$

Thus $\frac{3n}{n+2}$ is increasing.

Since $\frac{3n}{n+2}$ is bonded and increasing, then it has a supremum and infimum.

~~Since~~ When $n=1$; $\frac{3n}{n+2} = 1$, is in the set and is a lowerbound.

By theorem, 1 is a infimum

~~Since~~ Then claim: 3 is a supremum.

Since 3 is a upper bound for the set, then we need to prove

$\forall \varepsilon$, $\exists n$ s.t. $3 - \varepsilon < \frac{3n}{n+2} < 3$. Let's take $n > \frac{6}{\varepsilon} - 2$ (Archimedean property)

Then $\cancel{3-\varepsilon}$ $\frac{3n}{n+2} = 3 - \frac{6}{n+2} > 3 - \frac{6}{\frac{6}{\varepsilon}} = \cancel{3 - \frac{6\varepsilon}{2\varepsilon+6}} \cancel{\frac{3\varepsilon+9-3\varepsilon}{\varepsilon+3}}$
$= 3 - \varepsilon$

Thus we complete the proof.

Finally. $\inf = 1$, $\sup = 3$

2. (20 points) Determine if the following series are convergent or not

(a) $\sum_{n=1}^{\infty} n^4 e^{-n^2}$

(b) $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 17}$

(a) ~~Since~~ $\cancel{n^4 e^{-n^2} = \frac{n^4}{e^{n^2}}}$ ~~and~~ $\cancel{n \in \mathbb{N}}$, ~~then~~ $\cancel{\frac{n^4}{e^{n^2}} > 0}$.

~~Claim~~:

Lets use root-criteria.

Consider $\sqrt[n]{|a_n|} = \sqrt[n]{|n^4 \cdot e^{-n^2}|}$, since $n \in \mathbb{N}$. then $n^4 \cdot e^{-n^2} > 0$

Then $\sqrt[n]{|n^4 \cdot e^{-n^2}|} = \sqrt[n]{n^4 \cdot e^{-n^2}} = \sqrt[n]{n^4} \cdot \sqrt[n]{e^{-n^2}} = (\sqrt[n]{n})^4 \times (\sqrt[n]{e^{n^2}})^{-1}$

Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (\sqrt[n]{n})^4 \times \lim_{n\to\infty} (\sqrt[n]{e^{n^2}})^{-1}$

$= 1 \times (e^2)^{-1} = \frac{1}{e^2}$. ~~Since e~~

since $e > 1 \Rightarrow e^2 > 1$, then $\frac{1}{e^2} < 1$.

9 points

Thus $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$. Then the series $\sum_{n=1}^{\infty} n^4 \cdot e^{-n^2}$ is convergent

(b). ~~Since~~ $\cancel{\frac{2^n}{3^n - 17} = \frac{1}{\frac{3^n}{2^n} - \frac{17}{2^n}}}$

Lets use a proposition.

Since $\frac{2^n}{3^n - 17} \approx \frac{2^n}{3^n}$, then let compute $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = \lim_{n\to\infty} \frac{\left|\frac{2^n}{3^n - 17}\right|}{\left|\frac{2^n}{3^n}\right|}$

Since $n \in \mathbb{N}$. then $\frac{2^n}{3^n}$ and $\frac{2^n}{3^n - 17} > 0$

Then $\lim_{n\to\infty} \frac{\left|\frac{2^n}{3^n-17}\right|}{\left|\frac{2^n}{3^n}\right|} = \lim_{n\to\infty} \frac{3^n}{3^n - 17} = \lim_{n\to\infty} \frac{1}{1 - \frac{17}{3^n}} = \frac{1}{1 - \lim_{n\to\infty} \frac{17}{3^n}} = \frac{1}{1} = 1$

~~Thus~~ And since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is covergent, then by proposition $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 17}$ is convergent ✓

3. (20 points) Let (x_n) be a bounded sequence. For every $n \in \mathbb{N}$ define $a_n = \sup\{x_k : k \geq n\}$ and $b_n = \inf\{x_k : k \geq n\}$.

(a) Show that the sequences (a_n) and (b_n) are convergent. (Hint: show that they are monotonic)

(b) Let $a = \lim a_n$ and $b = \lim b_n$. Show that ~~$a \leq b$~~ $a \geq b$

(c) Show that (x_n) is convergent if and only if $a = b$.

(a) Since (x_n) is bounded, then $\forall n$, $|x_n| \leq M$

Thus x_k ~~$\geq n$~~, for $\forall k \geq n$ is also bounded $\Rightarrow$ a_n and b_n is bounded.

Thus by theorem, if we prove a_n and b_n is monotonic, then we can prove ~~it~~ they are convergent.

For (a_n), consider ~~a_n and a_{n+1}~~ $\forall n$ $a_{n+1} - a_n$ ~~and let $\sup\{x_k : k \geq n\} = r$~~

~~Let $\sup\{x_k : k \geq n+1\} = s$.~~

Then $a_{n+1} - a_n =$ ~~$s - r =$~~ $\sup\{x_k : k \geq n+1\} - \sup\{x_k : k \geq n\}$

~~If $x_n > x_{n+1}$, then $\sup r$~~ But we know $\sup\{x_k : k \geq n\} \geq \sup\{x_k : k \geq n+1\}$ Why?

Since $\sup\{x_k : k \geq n\} =$ ~~$\sup$~~ $\text{Max}\{\sup\{x_k : k \geq n+1\}, x_n\}$ ✓

Thus $a_{n+1} - a_n \leq 0 \Rightarrow (a_n)$ is monotonic decreasing

For (b_n), consider $\forall n$ ~~a_{n+1}~~ $b_{n+1} - b_n = \inf\{x_k : k \geq n+1\} - \inf\{x_k : k \geq n\}$

Since $\inf\{x_k : k \geq n\} = \text{Min}\{x_n, \inf\{x_k : k \geq n+1\}\}$, then $\inf\{x_k : k \geq n\} \leq \inf\{x_k : k \geq n+1\}$

Then $b_{n+1} - b_n \geq 0 \Rightarrow (b_n)$ is monotonic increasing

Finally (a_n) and (b_n) ~~is~~ are monotonic

By previous Analysis, (a_n) and (b_n) are convergent 6 pts

(b) Since $a_n = \sup\{x_k : k \geq n\}$, then $\forall$ $a_n \geq x_k$, for $k \geq n$.

Since $b_n =$ ~~$\sup$~~ $\inf\{x_k : k \geq n\}$, then $\forall$ $b_n \leq x_k$, for $k \geq n$ ✓

Thus $b_n \leq x_k \leq a_n$ for $\forall n$ s.t. $k \geq n$.

thus ~~$\Rightarrow$~~ $a_n \geq b_n$ $(\forall n)$ just apply sandwich lemma

Since $a = \lim a_n$, then $|a_n - a| < \varepsilon_1$ ($\forall \varepsilon_1 > 0, \exists n$) / Since $b = \lim b_n$, then $\forall \varepsilon_2 > 0$ $\exists n$ s.t. $|b_n - b| < \varepsilon_2$

then $-\varepsilon_1 + a < a_n < a + \varepsilon_1$ and $-\varepsilon_2 + b < b_n < b + \varepsilon_2$

Also since $a_n \geq b_n (\forall n)$, then $a+\varepsilon_1 > a_n \geq b_n > b-\varepsilon_2$

Thus we have $a+\varepsilon_1 > b-\varepsilon_2 \Rightarrow a-b > -\varepsilon_1-\varepsilon_2$.

Since $\varepsilon_1, \varepsilon_2 > 0$, then $-\varepsilon_1-\varepsilon_2 < 0$. This is not clear enough

then $a-b$ is bigger than a negative number.

then $a-b \geq 0 = a \geq b$ 6 pts

(c) $\Leftarrow$) if $a=b \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \Rightarrow \boxed{\lim_{n\to\infty} \sup\{x_k : k \geq n\} = \lim_{n\to\infty} \inf\{x_k : k \geq n\}}$ (*)

Suppose (x_n) is not convergent, then $\exists \varepsilon > 0. \forall n > n_0 : |x_n - L| \geq \varepsilon$.

$\Rightarrow x_n \geq \varepsilon + L$ or $x_n \leq L-\varepsilon$ $(\forall n > n_0)$ What is L?

~~Let X~~

$\Rightarrow \inf\{x_n : n \geq n_0\} \geq \varepsilon + L$ or $\sup\{x_n : n \geq n_0\} \leq L-\varepsilon$.

$\Rightarrow \inf\{x_k : k \geq n\} \geq \varepsilon + L$ or $\sup\{x_k : k \geq n\} \leq L-\varepsilon$

~~Thus Since~~

$\Rightarrow \lim_{n\to\infty} \inf\{x_k : k \geq n\} \geq L+\varepsilon$ or $\lim_{n\to\infty} \sup\{x_k : k \geq n\} \leq L-\varepsilon$

Since (*), then $L+\varepsilon \leq L-\varepsilon \Rightarrow \varepsilon \leq 0$. contradiction!

Thus (x_n) is convergent. This is not clear enough

$\Rightarrow$) Suppose $a \neq b$. since $a \geq b \Rightarrow a > b$.

Then $\lim_{n\to\infty} a_n > \lim_{n\to\infty} b_n \Rightarrow \lim_{n\to\infty} \sup\{x_k : k \geq n\} > \lim_{n\to\infty} \inf\{x_k : k \geq n\}$

$\Rightarrow \exists x_k : k \geq n$, such that $\inf\{x_k : k \geq n\} < x_k < \sup\{x_k : k \geq n\}$ when $n\to\infty$

Since (x_n) is convergent, then $\forall \varepsilon > 0, \exists n > n_0 : L-\varepsilon < x_n < L+\varepsilon$

Let $n=k$, $n_0 = n \Rightarrow \forall \varepsilon > 0, \exists k \geq n : L-\varepsilon \leq x_k \leq L+\varepsilon$

which means there is always x_k $(k \geq n)$ when $n\to\infty$, in the interval.

And $\inf\{x_k : k \geq n\}$, $\sup\{x_k : k \geq n\}$ also in the set.

Thus when $n\to\infty$, there always at 3 element satisfy condition. This is not clear enough

Thus there is limit 4 pts

Contradiction // Finally. we complete proof

4. (20 points) Let $\sum a_n$ be a convergent series. Determine if the following conclusions are true or false. Justify.

(a) $\sum a_n^2$ is convergent.

(b) $\sum |a_n|$ is convergent.

(c) $\sum |a_{n+1} - a_n|$ is convergent.

(a) ~~Let $a_n = \frac{}{n}$~~

~~Since~~ let $\sum a_n = \sum \frac{(-1)^n}{\sqrt{n}}$ which is convergent

Then $\sum a_n^2 = \sum \left(\frac{(-1)^n}{\sqrt{n}}\right)^2 = \sum \frac{1}{n}$ which is divergent.

False

7 pts

(b) ~~let~~ $\sum a_n = \sum \frac{(-1)^n}{n}$ is convergent.

6 pts

$\sum |a_n| = \sum \frac{1}{n}$ is divergent.

False

(c). Since ~~$\sum |a_{n+1} - a_n| = |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_{n+1} - a_n| + \cdots$~~

True ~~$= |\lim_{n\to\infty} a_n - a_1|$~~

✗

~~Suppose $\sum |a_{n+1} - a_n|$ is divergent, then $\lim_{n\to\infty} |a_{n+1} - a_n| \neq 0$~~

Since $\sum a_n$ is convergent, the $\lim_{n\to\infty} a_n = 0$.

Then $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = 0. \Rightarrow \lim_{n\to\infty} a_{n+1} - a_n = 0$

Since $\sum |a_{n+1} - a_n| < \sum |a_n|$, $\sum |a_n|$ is convergent

Thus $\sum$ is convergent

5. (20 points) Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Prove that $\lim_{x\to 0} f(x) = 0$ and that $\lim_{x\to a} f(x)$ does not exist for all $a \neq 0$.

To prove $\lim_{x\to 0} f(x)=0$. we need. $\forall \varepsilon>0 . \exists \delta, 0<|x|<\delta$ $\Rightarrow |f(x)|<\varepsilon$

- ~~Case~~ if $f(x)=0$. $(x\notin\mathbb{Q})$ ~~we are~~, $\varepsilon>0$, let $\varepsilon=\delta$ we are done.
- if $f(x)=x$. $(x\in\mathbb{Q})$. let $\delta=\varepsilon$, ~~we~~ $0<|x|<\delta \Rightarrow$ $0<|x|<\varepsilon \Rightarrow 0<|f(x)|<\varepsilon \Rightarrow \lim_{x\to 0} f(x)=0$.

12 pts

To prove $\lim_{x\to a} f(x)$ does not exist for all $a\neq 0$.

Use contradiction. Suppose $\lim_{x\to a} f(x)$ exists

- ~~That~~ $f(x)=x$, $(x\in\mathbb{Q}) \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} x = a$ This is not clear enough

Thus. $\forall\varepsilon>0, \exists\delta=\varepsilon, 0<|x-a|<\delta \Rightarrow |f(x)-a|<\varepsilon$.

Then $a-\varepsilon<x<a+\varepsilon \Rightarrow a-\varepsilon<f(x)<a+\varepsilon$

But there exists $x\in(a-\delta, a+\delta)$ but $x\notin\mathbb{Q}$ (density)

Then $f(x)=0$, then $a-\varepsilon<0<a+\varepsilon \Rightarrow a-\varepsilon<0<a+\varepsilon$

Then $x=0$. but $x\to a\neq 0$. contradiction.

- $f(x)=0$. $(x\notin\mathbb{Q}) \Rightarrow \lim_{x\to a} f(x) = \lim_{x\to a} 0 = 0$.

I do not understand

Thus $\forall\varepsilon>0$. $\exists\delta=\varepsilon$, $0<|x-a|<\delta \Rightarrow 0\ \varepsilon>0$.

5 pts

But there exists $x\in(a-\delta, a+\delta)$ but $x\in\mathbb{Q}$. (density)

Then $f(x)=x$, then $a-\delta<x<a+\delta \Rightarrow |f(x)|=|x|<\varepsilon$

Since $\delta=\varepsilon$, then $a-\varepsilon<x<a+\varepsilon \Rightarrow -\varepsilon<x<\varepsilon$

Then $a=0$. contradiction. // Finally, $\lim_{x\to a} f(x)$ does not exist for all $a\neq 0$