Workshop 9

Exercise 1

Study the parity of the following functions:

$$f_1(x) = x^2 + 2,\ f_2(x) = x^3 + 4,\ f_3(x) = e^x - e^{-x},\ f_4(x) = \frac{e^{2x} - 1}{e^{2x} + 1},\ f_5(x) = \frac{e^x}{(e^x + 1)^2}.$$

Exercise 2

Let $f, g : \mathbb{R} \to \mathbb{R}$ be odd functions. What about the parity of $f + g$, $f \times g$ and $f \circ g$?

Exercise 3

Let $f : \mathbb{R} \to \mathbb{R}$ be an even function. Assume that the restriction of f to $\mathbb{R}_-$ is increasing. What can be said about the monotonicity of the restriction of f to $\mathbb{R}_+$.

Exercise 4

(a) Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, and let $A = [-1, 4]$. Determine

  (i) the image of A by f;

  (ii) the pre-image of A by f.

(b) Consider the function $\sin : \mathbb{R} \to \mathbb{R}$. What is the image, by sin, of $\mathbb{R}$? Of $[0, 2\pi]$? Of $[0, \pi/2]$? What is the pre-image, by sin, of $[0, 1]$? Of $[3, 4]$? Of $[1, 2]$?

Exercise 5

Let $x \in \mathbb{R}_+$, be $f(x) = \frac{x}{x+1}$. Determine $f \circ f \circ \cdots \circ f(x)$ (where the symbol f appears n times) as a function of $n \in \mathbb{N}^*$ and $x \in \mathbb{R}_+$.

Exercise 6

Let $g : [0; +\infty[\to [0; 1[$ be defined by $g(x) = \frac{x}{1+x}$. Show that g is bijective and determine its inverse.

Exercise 7

Show that the function $f : \mathbb{R} \to \mathbb{R}_+^*$ defined by

$$f(x) = \frac{e^x + 2}{e^{-x}}$$

is bijective. Compute its inverse f^{-1}.

Exercise 8

Let E, F be two sets and $f : E \to F$. Let $A \subset E$ and $B \subset F$. Prove the equivalence:

$$f(A) \cap B = \varnothing \iff A \cap f^{-1}(B) = \varnothing.$$

Exercise 9 ($\star$)

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x/(1 + x^2)$.

(a) Is f injective? surjective?

(b) Show that $f(\mathbb{R}) = [-1, 1]$.

(c) Show that the restriction $g : [-1, 1] \to [-1, 1]$, $g(x) = f(x)$ is a bijection.

Exercise 10 ($\star$)

Let $f : X \to Y$. Show that the following conditions are equivalent:

(a) f is injective.

(b) For all subsets A, B of X, we have $f(A \cap B) = f(A) \cap f(B)$.

Exercise 11 ($\star$)

Let $f : \mathbb{Z} \times \mathbb{N}^* \to \mathbb{Q}$, $(p, q) \mapsto p + \frac{1}{q}$. Is f injective, surjective?