Workshop 8

Exercise 1

(a) Let f and g be the two functions from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f(x) = 3x + 1 \text{ and } g(x) = x^2 - 1.$$

Calculate $f \circ g$ and $g \circ f$.

(b) In the following examples, determine two functions u and v such that $h = u \circ v$:

$$h_1(x) = \sqrt{3x-1}, \quad h_2(x) = \sin\left(x + \frac{\pi}{2}\right), \quad h_3(x) = \frac{1}{x+7}.$$

Exercise 2

Let $f : I \to J$ be a function. Determine whether, from properties (a)-(d), we can deduce that:

- f is injective (but not necessarily surjective);
- f is surjective (but not necessarily injective);
- f is bijective;
- we cannot say anything about f.

(a) $\forall y \in J,\ f^{-1}(\{y\}) \neq \varnothing$;

(b) $\forall y \in J,\ f^{-1}(\{y\})$ contains at most one element;

(c) $\forall y \in f(I),\ f^{-1}(\{y\}) \neq \varnothing$;

(d) $\forall y \in f(I),\ f^{-1}(\{y\})$ contains at most one element;

Exercise 3

Are the following functions injective? surjectives? bijectives?

$$f_1 : \mathbb{Z} \to \mathbb{Z},\ n \mapsto 2n,\ \ f_2 : \mathbb{Z} \to \mathbb{Z},\ n \mapsto -n,$$

$$f_3 : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2,\ \ f_4 : \mathbb{R} \to \mathbb{R}_+,\ x \mapsto x^2,$$

$$f_5 : \mathbb{C} \to \mathbb{C}, z \mapsto z^2.$$

Exercise 4

Are the following functions injective, surjective, bijective?

(a) $f : \mathbb{N} \to \mathbb{N},\ n \mapsto n + 1$.

(b) $g : \mathbb{Z} \to \mathbb{Z},\ n \mapsto n + 1$.

(c) $h : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x + y, x - y)$.

Exercise 5

Is the map $f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto (x + y, xy)$ injective? surjective?

Exercise 6

Let f and g be the functions of $\mathbb{N}$ in $\mathbb{N}$ defined by $f(x) = 2x$ and

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ 0 & \text{if } x \text{ is odd.} \end{cases}$$

Determine $g \circ f$ and $f \circ g$. Are the functions f and g injective? surjective? bijective?

Exercise 7

Consider 4 sets A, B, C and D, and applications $f : A \to B$, $g : B \to C$ and $h : C \to D$. Show that

$$g \circ f \text{ injective} \implies f \text{ injective},$$

$$g \circ f \text{ surjective} \implies g \text{ surjective}.$$

Show that :

$$\big(g \circ f \text{ and } h \circ g \text{ are bijective }\big) \iff \big(f, g \text{ and } h \text{ are bijective}\big).$$

Exercise 8

Let E be a set. For $A \in \mathcal{P}(E)$ a subset of E, we denote $\bar{A}$ as its complement. Is the function $\phi : \mathcal{P}(E) \to \mathcal{P}(E)$, $A \mapsto \bar{A}$ injective? surjective?

Exercise 9 ($\star$)

Let $f : \mathbb{N}^2 \to \mathbb{N}^*$, $(n, p) \mapsto 2^n(2p + 1)$. Show that f is a bijection. Deduce a bijection of $\mathbb{N}^2$ onto $\mathbb{N}$.