Workshop 6

Exercise 1

We equip the set $E = \mathbb{R}^2$ with the relation $\mathcal{R}$ defined by

$$(x, y)\ \mathcal{R}\ (x', y') \iff \exists a > 0,\ \exists b > 0 \mid x' = ax \text{ and } y' = by.$$

(a) Show that $\mathcal{R}$ is an equivalence relation.

(b) Give the equivalence class of the elements $A = (1, 0)$, $B = (0, -1)$ and $C = (1, 1)$.

(c) Determine the equivalence classes of $\mathcal{R}$.

Exercise 2

Let E be a fixed set containing at least two elements. We consider the following binary relation on $\mathcal{F}(E, \mathbb{R}_+)$:

$$\forall (f, g) \in \mathcal{F}(E, \mathbb{R}_+) \times \mathcal{F}(E, \mathbb{R}_+), f \preccurlyeq g \iff [\forall x \in E, f(x) \leqslant g(x)]$$

(a) Show that $\preccurlyeq$ defines an order relation on $\mathcal{F}(E, \mathbb{R}_+)$.

(b) Is the order thus defined total?[1]

(c) Let $f \in \mathcal{F}(E, \mathbb{R}_+)$. Write the definitions of "f is upper bounded" and "$\{f\}$ is upper bounded".

(d) Show that, for this order, $\mathcal{F}(E, \mathbb{R}_+)$ has a smallest element by specifying it.

Exercise 3

We define the relation $\mathcal{R}$ on $\mathbb{N}^*$ by $p\mathcal{R}q \iff \exists k \in \mathbb{N}^*,\ q = p^k$. Show that $\mathcal{R}$ defines a partial order on $\mathbb{N}^*$. Determine the upper bounds of $\{2, 3\}$ for this order.

Exercise 4

We equip $\mathbb{R}^2$ with the relation denoted $\prec$ defined by

$$(x, y) \prec (x', y') \iff x \leq x' \text{ and } y \leq y'.$$

(a) Show that $\prec$ is an order relation on $\mathbb{R}^2$. Is the order total?

(b) Does the closed disk with center O and radius 1 have any upper bounds? a supremum? a maximal element?

Exercise 5

We define on $\mathbb{R}^2$ the relation $\prec$ by

$$(x, y) \prec (x', y') \iff \big((x < x') \text{ or } (x = x' \text{ and } y \leq y')\big).$$

Prove that this defines an order relation on $\mathbb{R}^2$.

Exercise 6 ($\star$)

Let E be an ordered set. Prove that every subset of E has a maximal element if and only if every increasing sequence of E is stationary.

Exercise 7 ($\star\star$)

An order $\leq$ on a set E is said to be well-founded if there is no strictly decreasing infinite sequence (x_n) of E. Show that $\mathbb{N}^2$ equipped with the lexicographic order (the order from Exersice 5) is well-founded.

[1] A total order is just another way of saying a linear order