Workshop 4

Exercise 1
Prove that no function exists from $\mathbb{N} \to \mathbb{N}$ such that $f(n) > f(n+1)$.

Exercise 2
Show that, for all $n \in \mathbb{N}^*$, we have $2^{n-1} \leq n! \leq n^n$.

Exercise 3
Prove by induction that, for all $n \in \mathbb{N}^*$, 6 divides $7^n - 1$.

Exercise 4
For $n \in \mathbb{N}$, we consider the following property:

$$P_n : \ 2^n > n^2.$$

1. Show that the implication $P_n \implies P_{n+1}$ is true for $n \geq 3$.
2. For what values of n is the property P_n true?

Exercise 5
Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by $u_0 = 2$, $u_1 = 3$ and, for all $n \in \mathbb{N}$, $u_{n+2} = 3u_{n+1} - 2u_n$. Show that, for all $n \in \mathbb{N}$, $u_n = 1 + 2^n$.

Exercise 6
Let $(u_n)_{n\in\mathbb{N}^*}$ be the sequence defined by $u_1 = 3$ and for all $n \geq 1$, $u_{n+1} = \frac{2}{n}\sum_{k=1}^{n} u_k$. Show that, for all $n \in \mathbb{N}^*$, we have $u_n = 3n$.

Exercise 7
Let (u_n) be the sequence defined by $u_0 = 1$ and, for all $n \geq 0$, $u_{n+1} = u_0 + u_1 + \cdots + u_n$. Show that, for all $n \geq 1$, $u_n = 2^{n-1}$.

Exercise 8
Prove that every convex polygon with n sides has $\frac{n(n-3)}{2}$ diagonals.

Exercise 9 ($\star$)
1. Write $\cos((n+1)°)$ as a function of $\cos(n°)$, $\cos(1°)$ and $\cos\big((n-1)°\big)$.
2. Prove that $\cos(1°)$ is irrational.

Exercise 10 ($\star$)
Let n be a strictly positive integer and p_n the n-th prime number. Show that there are infinitely many prime numbers. Show the following inequality: $p_n \leq 2^{2^n}$.

Exercise 11 ($\star\star$)
Show that, for any integer $n \geq 3$, we can find n strictly positive integers $x_1, \ldots, x_n$, two by two distinct, such that

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} = 1.$$

Exercise 12 ($\star\star$)
Let $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Show that $f(n) = n$ for all $n \in \mathbb{N}$. (Hint : Show by induction on n that $P_n\colon \forall k \geq n \implies f(k) \geq n$.)