Workshop 2

Exercise 1

Recall that $\sqrt{2}$ is an irrational number.

1. Prove that if a and b are two relative integers such that $a + b\sqrt{2} = 0$, then $a = b = 0$.

2. Deduce that if m, n, p and q are relative integers, then

$$m + n\sqrt{2} = p + q\sqrt{2} \iff (m = p \text{ and } n = q).$$

Exercise 2

Prove that if you store $(n+1)$ pairs of socks in n separate drawers, then there is at least one drawer containing at least 2 pairs of socks.

Exercise 3

Let $n \geq 1$ be a natural integer. We are given $n+1$ real numbers $x_0, x_1, \ldots, x_n$ of $[0,1]$ verifying $0 \leq x_0 \leq x_1 \leq \cdots \leq x_n \leq 1$. We want to demonstrate by contradiction the following property: there are two of these real numbers whose distance is less than or equal to $1/n$.

1. Write using quantifiers and the values $x_i - x_{i-1}$ a logical formula equivalent to the property.

2. Write the negation of this logical formula.

3. Write a proof by contradiction of the property (we can show that $x_n - x_0 > 1$).

4. Give a proof using the drawer principle.

Exercise 4

Let $a \in \mathbb{R}$. Show that : $\forall \varepsilon > 0, |a| \leq \varepsilon \implies a = 0$.

Exercise 5

Let a and b be two real numbers. Consider the following proposition: if $a + b$ is irrational, then a or b are irrational.

1. What is the contrapositive of this proposition?

2. Prove the proposition.

3. Is the converse of this proposition always true?

Exercise 6

Show that, for any relative integer n, $n(n-5)(n+5)$ is divisible by 3.

Exercise 7

The goal of the exercise is to demonstrate that the product of two integers that are not divisible by 3 is not divisible by 3.

1. Let n be an integer. What are the possible remainders in the Euclidean division of n by 3?

2. Deduce that if n is not divisible by 3, then n is written as $3k+1$ or $3k+2$, with k an integer. Is the converse true?

3. Let n be an integer written as $3k+1$ and m be an integer written as $3l+1$. Verify that

$$n \times m = 3(3kl + k + l) + 1.$$

   Deduce that $n \times m$ is not divisible by 3.

4. Demonstrate the property announced by the exercise.

Exercise 8

Determine the real numbers x such that $\sqrt{2-x} = x$.

Exercise 9

In this exercise, we want to determine all functions $f : \mathbb{R} \to \mathbb{R}$ verifying the following relation:

$$\forall x \in \mathbb{R},\ f(x) + xf(1-x) = 1 + x.$$

1. Consider f a function satisfying the previous relation. What is the value of $f(0)$? $f(1)$?
2. Let $x \in \mathbb{R}$. Substituting x by $1-x$ in the relation, determine $f(x)$.
3. What are the functions f that solve the problem?

Exercise 10

Let $f : \mathbb{R} \to \mathbb{R}$. Show that f can be uniquely written as the sum of an even function and the sum of an odd function.

Exercise 11 $(\star)$

Let a and $n \geq 2$ be two integers. Prove the following assertions.

1. If $a^n - 1$ is prime, then $a = 2$ and n is prime.
2. If $a^n + 1$ is prime, where $a \geq 2$ then n is even.
3. If $a^n + 1$ is prime, where $a \geq 2$ then a is even and n is a power of 2.

Exercise 12 $(\star)$

Let p be a prime number. Show that $\sqrt{p}$ is irrational. Generalize to $\sqrt{n}$ where n is not a perfect square.