Workshop 12

Exercise 1

Are the following sets countable?

(a) $\{2^n;\ n \geq 0\}$;

(b) $\mathbb{N} \times \mathbb{R}$;

(c) $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2};\ a, b \in \mathbb{Q}\}$;

(d) the set of prime numbers;

(e) the set of functions from $\mathbb{R}$ into $\mathbb{R}$.

Exercise 2

Let (f_n) be a sequence of applications of $\mathbb{N}$ into $\mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by: for all $n \in \mathbb{N}$, $f(n) = f_n(n) + 1$. Show that for any integer p, we have $f_p \neq f$. Deduce that the set of applications of $\mathbb{N}$ into $\mathbb{N}$ is not countable.

Exercise 3

(a) Show that $]0, 1[$ and $]0, 1]$ are in bijection (we can use a bijection from $]0, 1]$ to $]0, 1[$ which associates $1/(n + 1)$ with $1/n$.)

(b) Show that $]0, 1[$ and $[0, 1]$ are in bijection.

Exercise 4

Let $(I_\alpha)_{\alpha \in A}$ be a family of pairwise disjoint non-empty open intervals. Show that A is necessarily at most countable.

Exercise 5

A sequence of integers (n_k) is said to be stationary if there exists an integer $p \in \mathbb{N}$ such that, for all $k \geq p$, $n_k = n_p$. Prove that the set of sequences of almost zero integers and that the set of sequences of stationary integers are countable.

Exercise 6 ($\star$)

(a) Compute $\sum_{n \geq 0} \frac{1}{2^{n+1}}$.

(b) Let (u_n) be a sequence of $[0, 1]$. Show that, for all $n \geq 1$, there exists an element x_n in $[0, 1] \setminus \bigcup_{k=0}^{n} \left[u_k - \frac{1}{2^{k+2}}, u_k + \frac{1}{2^{k+2}}\right]$.

(c) Why can we extract from the sequence (x_n) a subsequence convergent to $\ell \in [0, 1]$?

(d) Prove that $[0, 1]$ is not countable.

Exercise 7 ($\star$)

We say that a real number x is an algebraic number if there exist $d \in \mathbb{N}^*$ and relative integers $a_0, \ldots, a_d$ with $a_d \neq 0$ such that

$$a_d x^d + \cdots + a_1 x + a_0 = 0.$$

The smallest integer d verifying this property is then the degree of x.

(a) What are the algebraic numbers of degree 1?

(b) Prove that the set of algebraic numbers of degree d is at most countable.

(c) Prove that the set of algebraic numbers is countable.

Exercise 8 ($\star$)

(a) Prove that the set of finite subsets of $\mathbb{N}$ is countable.

(b) Assume that the set of subsets of $\mathbb{N}$ is countable and denote by $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ a bijection. Considering $A = \{n \in \mathbb{N};\ n \notin f(n)\}$, find a contradiction.