Workshop 11

Exercise 1

We draw in a plane $n \geq 3$ lines in general position (that is to say that two lines are never parallel, and 3 lines are never concurrent). How many triangles have we thus drawn?

Exercise 2

A company has 18 employees, 8 of whom are women. For a survey, 3 people are chosen at random. What is the number of samples that include at least 2 men?

Exercise 3

Let A be the set of 7-digit numbers that do not contain any "1". Determine the number of elements in the following sets:

(a) A.

(b) A_1, set of numbers in A that have 7 different digits.

(c) A_2, set of even numbers in A.

(d) A_3, set of numbers in A whose digits form a strictly increasing sequence (in the order they are written).

Exercise 4

Consider a set X of $n+1$ (distinct) integers chosen from $\{1, \ldots, 2n\}$. Show that among the elements of X, we can always find 2 integers whose sum is $2n+1$.

Exercise 5

Solve the following two equations, with unknown $n \in \mathbb{N}$:

(a) $4\binom{n}{8} = \binom{n}{9}$, with $n \geq 9$.

(b) $\binom{3n}{1} + \binom{3n}{2} + \binom{3n}{3} = 115n$, with $n \geq 1$.

Exercise 6

Let E be the 12-element set $\{a, b, c, d, e, f, g, h, i, j, k, l\}$.

(a) Count the 5-element parts of E that contain

  (i) a and b;

  (ii) a but not b;

  (iii) b but not a;

  (iv) neither a nor b.

(b) Deduce the relation

$$\binom{12}{5} = \binom{10}{3} + 2\binom{10}{4} + \binom{10}{5}.$$

(c) Generalize the result obtained by proving, by counting, that for $2 \leq p \leq n$, we have

$$\binom{n}{p} = \binom{n-2}{p-2} + 2\binom{n-2}{p-1} + \binom{n-2}{p}.$$

(d) Find the previous result by applying Pascal's triangle formula: $\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$, for $1 \leq p \leq n-1$.

Exercise 7

Let $1 \leq p \leq n$. Consider n balls and two boxes A and B. A sample consists of one ball in box A and $p-1$ balls in box B. By counting these samples in two different ways, establish the formula

$$n\binom{n-1}{p-1} = p\binom{n}{p}.$$

Find this formula by a computation.

Exercise 8

A book has 14 chapters.

(a) How many ways are there to choose 3 chapters in this book?

(b) For $k = 3, \dots, 14$, count the choices of 3 chapters for which k is the largest number of the chosen chapters.

(c) Deduce that

$$\binom{14}{3} = \binom{13}{2} + \binom{12}{2} + \cdots + \binom{3}{2} + \binom{2}{2}.$$

(d) Generalize the previous counts to show that, for $1 \leq p \leq n$, we have

$$\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1}.$$

(e) Notice that for $k > p$ we have $\binom{k}{p} = \binom{k+1}{p+1} - \binom{k}{p+1}$. Use this to prove the previous formula.

Exercise 9 ($\star$)

Let p, q, m be natural integers, with $q \leq p \leq m$. Prove by counting that

$$\binom{m}{p} = \sum_{j=0}^{q} \binom{q}{j} \times \binom{m-q}{p-j}.$$

Exercise 10 ($\star$)

In my house, there is a staircase with 17 steps. To go down this staircase, I can go down one step at a time, down two steps, or down three steps at a time. How many ways are there to go down this staircase?

Exercise 11 ($\star$)

Let $n, p \geq 1$ be two integers.

(a) How many strictly increasing functions are there from $\{1, \dots, p\}$ to $\{1, \dots, n\}$?

(b) (i) Let $f : \{1, \dots, p\} \to \{1, \dots, n\}$ be an increasing function. Let $\phi(f)$ be the function defined on $\{1, \dots, p\}$, with values in $\{1, \dots, n+p-1\}$, by $\phi(f)(k) = f(k) + k - 1$. Prove that $\phi(f)$ is strictly increasing.

(ii) Let $g : \{1, \dots, p\} \to \{1, \dots, n+p-1\}$ be a strictly increasing function. Let $\psi(g)$ be the function defined on $\{1, \dots, p\}$, with values in $\{1, \dots, n\}$, by $\psi(g)(k) = g(k) - k + 1$. Show that $\psi(g)$ is increasing.

(iii) How many increasing functions are there from $\{1, \dots, p\}$ to $\{1, \dots, n\}$?