Workshop 10

Exercise 1

Prove that the set of functions from $\mathbb{N}$ to $\{0,1\}$ is in bijection with $\mathcal{P}(\mathbb{N})$.

Exercise 2

If E and F are two nonempty finite sets, then $\mathcal{F}(E,F)$ is a finite set and

$$\operatorname{Card}\mathcal{F}(E,F) = \operatorname{Card} F^{\operatorname{Card} E}.$$

Exercise 3

Let A be a set of cardinal a. Prove the cardinal of $\mathcal{P}(A)$ is 2^a.

Exercise 4

Let E_n be the set of polynomials of degree less than or equal to $n \in \mathbb{N}$ and whose coefficients are 0 or 1. Compute $\operatorname{Card} E_n$.

Exercise 5

Let E be a finite set and A be a subset of E, notice we have the following:

$$\operatorname{Card} A = \sum_{x\in E} \mathbb{1}_A(x).$$

(a) Shoat that if E is a finite set and $A \subset E$ then Card $A \leq$ Card E

(b) Show $\operatorname{Card} A = \operatorname{Card} E \iff A = E$.

Exercise 6

If E and F are two finite sets, then $E \times F$ is finite and $\operatorname{Card}(E \times F) = \operatorname{Card} E \operatorname{Card} F$.

Exercise 7

Let E and F be two finite sets, $f \in \mathcal{F}(E,F)$. Show the following:

(a) $\operatorname{Card} E \geq \operatorname{Card} f(E)$ and $\operatorname{Card} E = \operatorname{Card} f(E) \iff f$ is injective.

(b) If f is surjective, then $\operatorname{Card} E \geq \operatorname{Card} F$.

(c) If f is injective, then $\operatorname{Card} E \leq \operatorname{Card} F$.

(d) If f is bijective, then $\operatorname{Card} E = \operatorname{Card} F$.

Exercise 8

Let $A_1, A_2, \ldots, A_n$ be n sets (not necessarily disjoint). Prove that we have the following equality of cardinals:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\leqslant i_1<\cdots<i_k\leqslant n} |A_{i_1}\cap\cdots\cap A_{i_k}|\right)$$

Exercise 9 ($\star$)

Let X be a set. Construct an injection from X to $\mathcal{P}(X)$. Can we have a bijection between X and $\mathcal{P}(X)$?