Workshop 5

Exercise 1

Give a positive integer and a negative integer that are:

(a) congruent to 0 mod 5 and not congruent to 0 mod 6.

(b) congruent to 0 mod 5 and congruent to 0 mod 6.

(c) congruent to 2 mod 4 and congruent to 8 mod 6.

(d) congruent to 3 mod 4 and congruent to 3 mod 5.

(e) congruent to 1 mod 3 and congruent to 1 mod 7.

Exercise 2

Find all the integers n such that this equation is satisfied:

$$2n + 5 \equiv 3n + 1 \mod 3.$$

Exercise 3

State whether the following relations are reflexive, symmetric, transitive:

(a) $E = \mathbb{Z}$ and $x\mathcal{R}y \iff x = -y$;

(b) $E = \mathbb{R}$ and $x\mathcal{R}y \iff \cos^2 x + \sin^2 y = 1$;

(c) $E = \mathbb{N}$ and $x\mathcal{R}y \iff \exists p, q \geq 1,\ y = px^q$ (p and q are integers).

Which of the preceding examples are equivalence relations?

Exercise 4

Is the orthogonality relation between two lines in the plane symmetrical? reflexive? transitive?

Exercise 5

On $\mathbb{R}^2$, we define the equivalence relation $\mathcal{R}$ by

$$(x, y)\mathcal{R}(x', y') \iff x = x'.$$

Prove that $\mathcal{R}$ is an equivalence relation.
($\star$) Then determine the equivalence class of an element $(x_0, y_0) \in \mathbb{R}^2$.

Exercise 6

We define on $\mathbb{R}$ the relation $x\mathcal{R}y$ if and only if $x^2 - y^2 = x - y$.

(a) Show that $\mathcal{R}$ is an equivalence relation.

(b) ($\star$) Calculate the equivalence class of an element x of $\mathbb{R}$. How many elements are there in this class?

Exercise 7

Let E be a set. We define on $\mathcal{P}(E)$, the set of subsets of E, the following relation:

$$A\mathcal{R}B \text{ if } A = B \text{ or } A = \bar{B},$$

where $\bar{B}$ is the complement of B (in E). Prove that $\mathcal{R}$ is an equivalence relation.

Exercise 8

Let E be a non-empty set and $\alpha \subset \mathcal{P}(E)$ non-empty verifying the following property:

$$\forall X, Y \in \alpha,\ \exists Z \in \alpha, Z \subset (X \cap Y).$$

We define on $\mathcal{P}(E)$ the relation $\sim$ by $A \sim B \iff \exists X \in \alpha,\ X \cap A = X \cap B$. Prove that this defines an equivalence relation on $\mathcal{P}(E)$.
($\star$) What are the equivalence classes of $\varnothing$ and E?