Workshop 7

Exercise 1

Let E be a set and A and B two subsets of E. Of which sets are the following functions the characteristic functions?

(a) $\min(1_A, 1_B)$.

(b) $\max(1_A, 1_B)$.

(c) $1_A \cdot 1_B$.

(d) $1 - 1_A$.

(e) $1_A + 1_B - 1_A \cdot 1_B$.

(f) $(1_A - 1_B)^2$.

Exercise 2

Any finite subset of a totally ordered set $(E, \preccurlyeq)$ has a greatest element and a smallest element. Show that this result is false if the order is no longer total.

Exercise 3

Let $(E, \leqslant)$ be an ordered set. Let A be a nonempty subset of E having a smallest element and a greatest element.

(a) Show that $\min A \leqslant \max A$.

(b) What can we say about the about A if $\min A = \max A$?

Exercise 4

Let E be a nonempty set. Consider the order induced by inclusion on $\mathcal{P}(E)$. Let $\mathcal{A}$ be a subset of $\mathcal{P}(E)$.

(a) Show that $\bigcap_{A \in \mathcal{A}} A$ lowers $\mathcal{A}$.

(b) Deduce that $\mathcal{A}$ has a lower bound in $(\mathcal{P}(E), \subset)$.

(c) Show that $\mathcal{A}$ has an upper bound in $(\mathcal{P}(E), \subset)$.

Exercise 5

Let $(E, \leqslant)$ be an ordered set. Show that if A and B are two subsets of E such that $\sup(A), \sup(B)$ and $\sup\{\sup(A), \sup(B)\}$ exist, then $\sup(A \cup B)$ exists and calculate it (we can try on examples to get an idea of the result to prove.)

Exercise 6

(a) Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, and let $A = [-1, 4]$. Determine

  (i) the direct image of A by f;

  (ii) the inverse image[1] of A by f.

(b) Consider the function $\sin : \mathbb{R} \to \mathbb{R}$. What is the direct image, by sin, of $\mathbb{R}$? Of $[0, 2\pi]$? Of $[0, \pi/2]$? What is the inverse image, by sin, of $[0, 1]$? Of $[3, 4]$? Of $[1, 2]$?

Exercise 7 ($\star$)

Let E and F be two sets and $f : E \to F$. Prove that

(a) $\forall A \in \mathcal{P}(E)$, $A \subset f^{-1}(f(A))$.

(b) $\forall B \in \mathcal{P}(F)$, $f(f^{-1}(B)) \subset B$.

[1] Same thing as a pre-image

(c) ($\star$) Does equality hold in general?

Exercise 8 ($\star$)

[$\star$] Let E and F be two sets and let $f : E \to F$. Take A and B two subsets of F.

(a) Prove that $A \subset B \implies f^{-1}(A) \subset f^{-1}(B)$. Is the converse true?

(b) Prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) Prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Exercise 9 ($\star$)

Let E, F be two sets and $f : E \to F$. Let $A \subset E$ and $B \subset F$. Prove the equivalence:

$$f(A) \cap B = \varnothing \iff A \cap f^{-1}(B) = \varnothing.$$