HOMEWORK 7

Exercise 1. (30 points) Recall that a set C is infinite if C is not finite. Prove the following statements.

(a) Let C, D be sets. If C is infinite and $C \subseteq D$, then D is infinite.

(b) If C is an infinite set and $C = A \cup B$, then at least one of the sets A or B is infinite.

(c) Suppose A is a set and p is an object not in A. If $A \approx A \cup \{p\}$, then A is infinite.

(d) Prove that if A is finite and B is infinite, then $B - A$ is infinite.

Exercise 2. (20 points) In each of the following cases, write a bijective function $f : A \to B$ and its inverse $g : B \to A$. Verify that g is indeed the inverse of f.

(a) $A = [n] \times [m]$ and $B = [mn]$.

(b) $A = [9]_0 \times [9]_0 \times [9]_0$ and $B = [999]_0$. Here $[n]_0 := [n] \cup \{0\}$.

Exercise 3. (20 points) Let $f : A \to B$ be a function between non-empty finite sets such that all the sets $f^{-1}(b)$, for $b \in B$, have the same cardinality. Prove that $|B|$ divides $|A|$.

Exercise 4. (30 points) Use the pegeonhole principle to prove the following facts.

(a) Among any 6 integers, at least two of them are congruent modulo 5.

(b) If we place 5 points inside a unit square, there are at least two of them whose distance is at most $\frac{\sqrt{2}}{2}$.