Exercise 1. (25 points) Let $f : A \to B$ and $g : C \to D$ be functions. Define a relation $f \times g \subset (A \times C) \times (B \times D)$ as $f \times g = \{((a, c), (b, d)) \, : \, f(a) = b, \ g(b) = d\}$.

(a) Prove that $f \times g$ is a function from $A \times C$ to $B \times D$ and compute

(b) Prove that the function $h : \mathbb{N}^2 \to \mathbb{N}^2$ given by $h(m, n) = (m^2, n+1)$ is of the form $f \times g$ for some functions $f, g : \mathbb{N} \to \mathbb{N}$.

(c) Give an example of a function $h : \mathbb{N}^2 \to \mathbb{N}^2$ that is not of the form $f \times g$.

Exercise 2. (25 points) Let

$$f(x) = \begin{cases} 2x+3 & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases} \quad , \quad g(x) = \begin{cases} 7-2x & \text{if } x \leq 2 \\ x+1 & \text{if } x > 2 \end{cases}$$

(a) Which one is injective, which one is surjective?, which one is bijective? Justify

(b) Compute $g \circ f$ and $f \circ g$.

Exercise 3. (25 points) For each of the following functions $f : A \to \mathbb{R}$, show that they are one to one, compute $\mathrm{Im}(f)$, and find a formula for the inverse $f^{-1} : \mathrm{Im}(f) \to A$.

(a) $f : (-2, \infty) \to \mathbb{R}$ given by $f(x) = \dfrac{4x}{x+2}$.

(b) $f : (3, \infty) \to \mathbb{R}$ given by $\dfrac{5(x-1)}{x-3}$.

Exercise 4. (25 points) Decide if the following sentences are true or false. When they are true, give a proof. When they are false, give a counterexample.

(a) Every function $f : A \to A$ is injective or surjective.

(b) If $f : A \to A$ satisfies $f^2 = \mathrm{id}_A$ then f is bijective.

(c) If $f : A \to B$ and $g : B \to A$ are functions such that $g \circ f = \mathrm{id}_A$ then f is bijective.

(d) There is a bijective function $f : \mathbb{N} \to \mathbb{Z}$.