Student Feedback Report

Student Id	999027873
Exam Id	202411181040025
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Course Id	202401-104002-5
Course Name	BASIC CONCEPTS IN MATHEMATICS - Mid
Lecturer	Diego Armando SULCA

Open question score	Original Exam Grade	Final Exam Grade
79.00	79.00	**79.00**

Summary

Question number	Actual points	Max points
1	10.00	20.00
2	20.00	20.00
3	20.00	20.00
4.1	7.00	7.00
4.2	6.00	7.00
4.3	6.00	6.00
5.1	7.00	7.00
5.2	3.00	7.00
5.3	0	6.00

✓ Correct answer ✓ Partial answer × Incorrect answer ⦸ Not answered

(64)
ID 999027873
Exam 202411181040025

Basic Concepts in Mathematics 104002

Midterm
November 18, 2024

Your ID Number: 999027873

Your Name: Yue SHi

Guidelines

- **Duration: 2 hours.** Use of calculators, personal dictionaries, electronic devices, reference materials, personal notes or any other extra material is not allowed.
- Show all your work. Explain your solutions, quote theorems you are using.
  No credit will be given for non-justified answers!
  Write clear and complete answers for each problem in the same page.

1. Let P be a proposition, let $Q(x)$ be an open proposition and let U be a universe. Prove that

$$(P \Rightarrow (\forall x \in U)Q(x)) \equiv (\forall x \in U)(P \Rightarrow Q(x))$$

($\equiv$ means "is equivalent to").

We need to prove two directions

First, we prove $(P \Rightarrow (\forall x \in U) Q(x)) \Rightarrow (\forall x \in U)(P \Rightarrow Q(x))$

Then we have $P \Rightarrow (\forall x \in U) Q(x)$

Since $(\forall x \in U) Q(x)$, then for all $x \in U$, $Q(x)$ is true, then $Q(x)$ is always true.

Truth table

P	$(\forall x \in U)$ $Q(x)$	$P \Rightarrow (\forall x \in U) Q(x)$	$Q(x)$	$(\forall x \in U)$ $P \Rightarrow Q(x)$
T	T	T	T	T
T	T	T	F	T
F	T	T	T	T
F	T	T	F	T

~~Since $P \Rightarrow (\forall x \in U) Q(x)$, then and we know $(\forall x \in U) Q(x)$ means~~

~~$Q(x)$~~ is always true (since U is universe), then

Then ~~if P is true, then true $\Rightarrow$~~ true

Proof:

Since $Q(x)$ is an open proposition and we have $(\forall x \in U) Q(x)$ where U is a universe then we know $Q(x)$ is always true.

~~Thus $(P \Rightarrow (\forall x \in U) Q(x)) \equiv P \Rightarrow Q(x)$~~

Also since P is a proposition and ~~its~~ true or false don't depend on which x we take.

Thus $(P \Rightarrow (\forall x \in U) Q(x)) \equiv (\forall x \in U)(P \Rightarrow Q(x))$

See Next Page

Since Q(X) is an open proposition thus for ~~some~~ $X \in A \subseteq U$. Q(X) is true and for $X \in B \subseteq U$. Q(X) is false $(A \cup B = U)$

Since $(\forall x \in U)\,Q(X)$, thus for $x \in U$, Q(X) is true by definition of quantofier.

The quantified proposition can be true or false, you don`t have to assume that it is true

Thus $A = U$, $B = \phi$. Then Q(X) is always true whatever we take x.

~~Thus if P~~

Since P is a proposition. then it has only definitly true or false ~~that do don't depend on which x we take~~.

For $(P \Rightarrow (\forall x \in U)\,Q(X))$

Then, If P is true, we have [true ⇒ true] is true.

If P is false, we have [false ⇒ true] is true.

Then for $(\forall x \in U)(P \Rightarrow Q(X))$

~~Since~~ If P is true, then Q(X) can be true or false.

let $P \Rightarrow Q(X) = M(X)$, { if Q(X) is true. Then $(\forall x \in U)\,M(X)$ is true

if Q(X) is false then ~~$(\forall x \in U)$~~ M(X) is false. ~~But~~ contradicts to $(\forall x \in U)\,M(X)$. $(\forall x \in U)\,M(X)$ is true

It is not ok to say that a for all sentence is true by just looking to what happen with just one x

Thus Q(X) can not be false. thus if P is true then $(\forall x \in U)(P \Rightarrow Q(X))$ is true.

If P is false. $P \Rightarrow Q(X)$ is always ~~fa~~ true. Thus $(\forall x \in U)(P \Rightarrow Q(X))$ is always true.

Thus they are equivalent

This part is ok

Finally ~~if P is~~ $P \Rightarrow (\forall x \in U)\,Q(X)$ and $(\forall x \in U)(P \Rightarrow Q(X))$ is always true.

10
(1)

20
(2)

2. Prove the following equality for all $n \in \mathbb{N}$.

$$\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{n!2^n}$$

we use induction to prove it.

Basic step. when $n=1$, $\prod_{i=1}^{n}(2i-1) = \prod_{i=1}^{1}(2i-1) = 2-1 = 1$

$$\frac{(2n)!}{n!2^n} = \frac{2!}{1!\,2^1} = \frac{2}{1\times 2} = 1$$

Thus when $n=1$, $\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{n!2^n}$

Inductive step. suppose it is true for n, then $\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{n!\,2^n}$ (Hypothesis)

~~wh~~

Then for $n+1$, we have $\prod_{i=1}^{n+1}(2i-1) = \left[\prod_{i=1}^{n}(2i-1)\right](2(n+1)-1)$

~~Sn~~

$$= \left[\prod_{i=1}^{n}(2i-1)\right](2n+1)$$

Then we use hypothesis here.

$$= \frac{(2n)!}{n!\,2^n} \times (2n+1) = \frac{(2n)! \times (2n+1)(2n+2)}{(2n+2)\,n!\,2^n} = \frac{(2(n+1))!}{2n\cdot n!\cdot 2^n + 2n!\cdot 2^n}$$

$$= \frac{(2(n+1))!}{n\cdot n!\,2^{n+1} + n!\cdot 2^{n+1}} = \frac{(2(n+1))!}{2^{n+1}\cdot n!\,(n+1)} = \frac{(2(n+1))!}{2^{n+1}\cdot (n+1)!} = \frac{(2(n+1))!}{(n+1)!\,2^{n+1}}$$

Thus it is also true for $n+1$.

Since if the equality is true for n, the we can imply that it is also true for $n+1$.

Thus the equality is true for all $n \in \mathbb{N}$

3. (20 points) Define a sequence (a_n) as follows:

$$a_1 = 3, \quad a_2 = 9, \quad \text{and} \quad a_{n+1} = 5a_n - 6a_{n-1}, \ \forall n \geq 2$$

Prove that $a_n = 3^n$ for all n.

Use strong induction.

Basic step: Since $n \geq 2$. then when $n = 3$
since $a_{n+1} = 5a_n - 6a_{n-1}$, then $a_3 = 5a_2 - 6a_1$

Since $a_1 = 3$, $a_2 = 9$ then $a_3 = 27$
By $a_n = 3^n$, we know $a_3 = 3^3 = 27$. Thus $a_n = 3^n$ is true for $n = 2$

Inductive step: Suppose it is true for $3 \leq k \leq n$, $k \in \mathbb{Z}$
then $a_k = 3^k$. (Hypothesis) for $3 \leq k \leq n$, $k \in \mathbb{Z}$

why don`t 1,2?

Then for n+1,
we need to prove $a_{n+1} = 3^{n+1}$

Since $a_{n+1} = 5a_n - 6a_{n-1}$, then we use hypothesis here to substitute a_n and a_{n-1}.

Then $a_{n+1} = 5 \times 3^n - 6 \times 3^{n-1} = 15 \times 3^{n-1} - 6 \times 3^{n-1} = 9 \times 3^{n-1} = 3^{n+1}$.

Thus it is true for n+1.

Since if $a_n = 3^n$ is true for $3 \leq k \leq n$, $k \in \mathbb{Z}$, we can imply it is true for n+1.

Thus $a_n = 3^n$ is true for $n \geq 3$. (Basic step + Inductive step)

Note: we also need to check $n = 1$ and $n = 2$
when $n = 1$, $a_1 = 3$ and $a_1 = 3^1 = 3$. true
$n = 2$, $a_2 = 9$, and $a_2 = 3^2 = 9$. true.

Finally, $a_n = 3^n$ is true for all n.

20
(3)

4. (20 points) Let R be the relation on the set of real numbers $\mathbb{R}$ given by xRy if and only if $x - y \in \mathbb{Z}$.

(a) Prove that R is an equivalence relation.

(b) Prove that any $x \in \mathbb{R}$ is equivalent to one and only one number in the half-open interval $[0, 1)$.

(c) Is (a) true if we replace $\mathbb{Z}$ by the set $\{-2024, -2023 - 2022, \ldots, 2022, 2023, 2024\}$?

(a) To prove R is an equivalence relation
we need to check three things

1. Refflexivity: since $x - x = 0 \in \mathbb{Z}$, then by definition of R.
we have xRx is true ✓

2. Symmetry: Since if $x-y \in \mathbb{Z}$. then $-(x-y) \in \mathbb{Z}$, then $y-x \in \mathbb{Z}$
Thus ~~we have~~ [if xRy, then yRx.] is true ✓

3. Transitivity: Since if $x-y \in \mathbb{Z}$. and $y-z \in \mathbb{Z}$, then we know
$x-y=m$, $m \in \mathbb{Z}$. and $y-z=n$, $n \in \mathbb{Z}$.
Then $x=m+y$ and $z=y-n \Rightarrow$. $x-z=m+y-y+n=m+n$
Since $m+n \in \mathbb{Z}$. then $x-z \in \mathbb{Z}$
thus [if xRy, yRz, then xRz] is true. ✓

Thus three propert~~y~~ies are satisfied. Therefore R is an equivalence relation

✓ (4.1)

(b) let's take any $x \in \mathbb{R}$. since ~~x~~ $y \in \mathbb{R}$. by definition of R

First. ~~ext~~ existence,
Take any $x \in \mathbb{R}$. we need to find $y \in [0,1)$ such that xRy
thus ~~we~~ we need to find $y \in [0,1)$ such $x-y \in \mathbb{Z}$
~~If $y \in [0,1)$, we are done~~. Thus consider $y = x - [x]$.
~~If $y \notin [0,1)$, then~~ Let's check it works
$x-y = x - x + [x] = [x] \in [0,1)$ ✓.

Then, uniqueness.

Suppose we can find ~~z two~~ $y \in [0,1)$, and $y' \in [0,1)$

then we have $x-y \in \mathbb{Z}$ and $x-y' \in \mathbb{Z}$

Then ~~x~~ $x-y=m$ $(m \in \mathbb{Z})$ and $x-y'=n$ $(n \in \mathbb{Z})$ ($m \neq n$ since $y \neq y'$)

$y = x-m$ and $y' = x-n$.

so you assume y-y` positive

~~Since~~ Then $y-y' = -m+n \geq 1$. since $m, n \in \mathbb{Z}$

6 (4.2)

Thus $y \geq 1+y'$. contradiction.

Finally, there is a unique $y \in [0,1)$ such that xRy

(c) $xRy \Leftrightarrow x-y \in \{-2024, \dots, 2024\}$, $x, y \in \mathbb{R}$.

1. Reflexivity: $x-x = 0 \in \{-2024, \dots, 2024\}$. thus xRx ✓

2. Symmetry: If $x-y \in \{-2024, \dots, 2024\}$, then $y-x \in \{-(-2024), \dots, -2024\}$ $= \{-2024, \dots, 2024\}$.

Thus $xRy \Rightarrow yRx$

✓

3. Transitivity. If $x-y \in \{-2024, \dots, 2024\}$ and $y-z \in \{-2024, \dots, 2024\}$,

Then $x-y = m \in \{-2024, \dots, 2024\}$ and $y-z = n \in \{-2024, \dots, 2024\}$

then $x = m+y$ and $z = y-n$.

then $x-z = m+y-y+n = m+n$. which ~~is~~ can not be in $\{-2024, \dots, 2024\}$

Counterexample: $x-y = 2024$, $y-z = 2024$.

But $x-z = 4048 \notin \{-2024, \dots, 2024\}$.

✓ (4.3)

Thus it is not ~~tree~~ true

5. (20 points) Decide if the following sentences are true or false. When a sentence is true, give a proof. When a sentence is false, give a counterexample.

(a) If A, B are sets such that $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

(b) If A, B, C are sets such that $A \times B = A \times C$ then $B = C$.

(c) If $I_1, I_2, I_3, \ldots$ are subsets of $\mathbb{R}$ such that $\cap_{i=1}^{n} I_i \neq \emptyset$ for all n, then $\cap_{i=1}^{\infty} I_i \neq \emptyset$.

7 (5.1)

True

Very good!

(a) Since $P(A) = P(B)$, then by definition. for any set $X \in P(A)$: $X \subseteq A$ and for any set $X \in P(B)$: $X \subseteq B$.

~~Since $P(A) = P(B)$, let's take any set $X \in P(A)$ and also $X \in P(B)$~~

~~Then we have $X \subseteq A$ and $X \subseteq B$ for any set X~~

$\subseteq$) Let's take any $x \in A$, then $\{x\} \subseteq A$, then $\{x\} \in P(A)$

Since $P(A) = P(B)$, then $\{x\} \in P(B) \Rightarrow \{x\} \subseteq B \Rightarrow x \in B$, thus $A \subseteq B$

$\supseteq$) Let's take any $x \in B$, then $\{x\} \subseteq B$, then $\{x\} \in P(B)$

Since $P(A) = P(B)$ then $\{x\} \in P(A) \Rightarrow \{x\} \subseteq A \Rightarrow x \in A$, thus $B \subseteq A$

Thus $A = B$, True!

(b) True Since we have $A \times B$, by definition: $A \times B = \{(a,b) :$ for any $a \in A$ and $b \in B\}$

Also $A \times C = \{(a,c) :$ for any $a \in A$ and $c \in C\}$

Since $A \times B = A \times C$, then for any $b \in B$ ~~$\Rightarrow b \in C$. and for any $c \in C$ $c \in B$.~~ we have $b \in C$, thus $B \subseteq C$ and for any $c \in C$, we have $c \in B$, thus $C \subseteq B$

you need to organize your writing.
Let b in B. Then there is a such that (a,b) in AxB=AxC. Hence b in C........
Besides, you forgot the case A=empty set

3 (5.2)

Thus $B = C$ True!

True ~~False~~

(c) Since $\bigcap_{i=1}^{n} I_i \neq \emptyset$, then we know there exists at least one ~~some~~ element x in $\bigcap_{i=1}^{n} I_i$

Then there exists ~~some elements~~ x in $\{x \in I_i :$ for all ~~some~~ $i \in [1, n]\}$. for all n.

~~But if $x \notin I_{n+1}$, then $x \notin \{x \in I_i :$ for all $i \in [1, n+1]\}$.~~

~~Also $x \notin \{x \in I_i :$ for all $i \in [1, \infty)\}$. Thus the sentence is false~~

this is not a proof

~~Counterexample: $2 \in [2,3] \cap [-1,2] = \bigcap_{i=1}^{2} I_i \neq \emptyset$, but $2 \notin [2,3] \cap [-1,2] \cap [-1,1] = \bigcap_{i=1}^{3} I_i = \emptyset$~~

10

~~Also $\phi \supseteq \bigcap_{i=1}^{\infty} I_i$, thus~~

~~Suppose $\bigcap_{i=1}^{\infty} I_i = \phi$, then $x \notin \bigcap_{i=1}^{\infty} I_i$~~

~~Consider set $A = \{ x \in \bigcap_{i=1}^{n} I_i$ s.t. $x \notin \bigcap_{i=1}^{n} I_i \}$ for all n~~

~~We need to prove $A = \phi$, suppose $A \neq \phi$.~~

~~Then there exists the first element~~

Suppose $\bigcap_{i=1}^{\infty} I_i = \phi$. which means exists $x \in \bigcap_{i=1}^{k} I_i$ but (for some k∈n)

$x \notin \bigcap_{i=1}^{k+1} I_i$ why?, which means $x \notin I_{k+1}$

However, since $k \in n$, then $k+1 \in n$. thus $x \in \bigcap_{i=1}^{k+1} I_i \Rightarrow x \in I_{k+1}$

why?

Contradiction.

Thus $\bigcap_{i=1}^{\infty} I_i \neq \phi$

this is not a proof

✗ (5.3)