Basic Concepts in Mathematics 104002

Midterm

November 18, 2024

Your ID Number: | | | | | | | | | |

Your Name: ____________________

Guidelines

- **Duration: 2 hours.** Use of calculators, personal dictionaries, electronic devices, reference materials, personal notes or any other extra material is not allowed.
- Show all your work. Explain your solutions, quote theorems you are using.
  No credit will be given for non-justified answers!
  Write clear and complete answers for each problem in the same page.

1. Let P be a proposition, let $Q(x)$ be an open proposition and let U be a universe. Prove that

$$(P \Rightarrow (\forall x \in U)Q(x)) \equiv (\forall x \in U)(P \Rightarrow Q(x))$$

($\equiv$ means "is equivalent to").

Solution one: (true table + some argument with quantifiers)

Case 1: Suppose that $(\forall x \in U)Q(x)$ is true. Then $P \Rightarrow (\forall x \in U)Q(x)$ is true. In addition, given $x \in U$, $Q(x)$ is true and therefore $P \Rightarrow Q(x)$ is true, so, $(\forall x)(P \Rightarrow Q(x))$ is true. Hence $P \Rightarrow (\forall x \in U)Q(x)$ and $(\forall x)(P \Rightarrow Q(x))$ have the same true value in this case.

Case 2: Suppose that $(\forall x \in U)Q(x)$ is false. Then there is some $x_0 \in U$ such that $Q(x_0)$ is false.

Case 2.1: If P is true then $P \Rightarrow (\forall x \in U)Q(x)$ is false because $(\forall x \in U)Q(x)$ is false. In addition, $P \Rightarrow Q(x_0)$ is false because $Q(x_0)$ is false. Hence, $(\forall x)(P \Rightarrow Q(x))$ is false. Thus, $P \Rightarrow (\forall x \in U)Q(x)$ and $(\forall x)(P \Rightarrow Q(x))$ have the same true value in this case.

Case 2.2: If P is false, then, since a conditional with a false antecedent is always true, $P \Rightarrow (\forall x \in U)Q(x)$ is true, and $P \Rightarrow Q(x)$ is true for all x. The latter means that $(\forall x)(P \Rightarrow Q(x))$ is true. Hence, $P \Rightarrow (\forall x \in U)Q(x)$ and $(\forall x)(P \Rightarrow Q(x))$ have the same true value in this case too.

Solution two:

We are going to prove that $P \Rightarrow (\forall x \in U)Q(x)$ is true if and only if $(\forall x \in U)(P \Rightarrow Q(x))$ is true.

Assume that $P \Rightarrow (\forall x \in U)Q(x)$ is true. Given an arbitrary $x_0 \in U$ we want to show that $P \Rightarrow Q(x_0)$ is true. Suppose that P is true. Since $P \Rightarrow (\forall x \in U)Q(x)$ is true, $(\forall x \in U)Q(x)$ is also true, so in particular $Q(x_0)$ is true, as we wanted.

Assume now that $(\forall x \in U)(P \Rightarrow Q(x))$ is true. We prove that $P \Rightarrow (\forall x \in U)Q(x)$ is true. Assume that P is true and let us prove that $(\forall x \in U)Q(x)$ is true. Let $x \in U$. By the assumption $P \Rightarrow Q(x)$ is true, and since P is also true, $Q(x)$ must be true as well. Thus, $(\forall x \in U)Q(x)$ is true as we wanted.

Solution three: We are going to use the following general properties: a) $\sim (p \Rightarrow q) \equiv p \wedge \sim q$, b) $p \equiv q$ if and only if $\sim p \equiv \sim q$, and c) $\sim (\forall x \in U)p(x) \equiv (\exists x_0 \in U) \sim p(x_0)$.

On the one hand, $\sim (P \Rightarrow (\forall x \in U)Q(x)) \equiv P \wedge (\exists x_0 \in U) \sim Q(x_0)$, which is true if and only if P is true and $Q(x_0)$ is false for some x_0.

On the other hand, $\sim ((\forall x \in U)(P \Rightarrow Q(x))) \equiv (\exists x_0 \in U)(P \wedge \sim Q(x_0)$, which is true if and only if for some $x_0 \in U$, P is true and $Q(x_0)$ is false, or in other words, P is true and $Q(x_0)$ is false for some $x_0 \in U$.

Hence $\sim (P \Rightarrow (\forall x \in U)Q(x)) \equiv \sim ((\forall x \in U)(P \Rightarrow Q(x)))$ and therefore $P \Rightarrow (\forall x \in U)Q(x) \equiv (\forall x \in U)(P \Rightarrow Q(x))$.

Solution four: like solution three but more straightfoward. $P \Rightarrow (\forall x \in U)Q(x)$ is false if and only if P is true and $(\forall x \in U)Q(x)$ is false, that is, if and only if P is true and there is $x_0 \in U$ such that $Q(x_0)$ is false, that is, if and only if for some $x_0 \in U$, $P \Rightarrow Q(x_0)$ is false, that is, if and only if $(\forall x \in U)(P \Rightarrow Q(x_0))$ is false.

2. Prove the following equality for all $n \in \mathbb{N}$.

$$\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{n!2^n}$$

3. (20 points) Define a sequence (a_n) as follows:

$$a_1 = 3, \quad a_2 = 9, \quad \text{and} \quad a_{n+1} = 5a_n - 6a_{n-1}, \ \forall n \geq 2$$

Prove that $a_n = 3^n$ for all n.

4. (20 points) Let R be the relation on the set of real numbers $\mathbb{R}$ given by xRy if and only if $x-y \in \mathbb{Z}$.

(a) Prove that R is an equivalence relation.

(b) Prove that any $x \in \mathbb{R}$ is equivalent to one and only one number in the half-open interval $[0,1)$.

(c) Is (a) true if we replace $\mathbb{Z}$ by the set $\{-2024, -2023 - 2022, \ldots, 2022, 2023, 2024\}$?

Solution:

b) Existence: Given $x \in \mathbb{R}$ let $\lfloor x \rfloor$ be the largest integer below x. Then $x = \lfloor x \rfloor + y$ for some $y \geq 0$. Moreover $y < 1$, otherwise $\lfloor x \rfloor + 1$ would be an integer larger than $\lfloor x \rfloor$ below x. Since $x - y = \lfloor x \rfloor \in \mathbb{Z}$ we have xRy with $y \in [0,1)$.

Uniqueness: Given $x \in \mathbb{R}$, suppose there are $0 \leq y \leq y' < 1$ such that xRy and xRy'. By symmetriy and transitivity we also have yRy'. Hence $y' - y \in \mathbb{Z}$. Moreover, $0 \leq y' - y < 1$ because $0 \leq y \leq y' < 1$. Hence $y = y'$.

5. (20 points) Decide if the following sentences are true or false. When a sentence is true, give a proof. When a sentence is false, give a counterexample.

(a) If A, B are sets such that $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

(b) If A, B, C are sets such that $A \times B = A \times C$ then $B = C$.

(c) If $I_1, I_2, I_3, \ldots$ are subsets of $\mathbb{R}$ such that $\cap_{i=1}^{n} I_i \neq \emptyset$ for all n, then $\cap_{i=1}^{\infty} I_i \neq \emptyset$.

(b)
False: Take $A = \emptyset$, $B = \{1\}$ and $C = \{2\}$. Then $A \times B = A \times C = \emptyset$, however $B \neq C$.
(c)
False: Take $I_n = (0, \frac{1}{n})$. Then $I_1 \cap \cdots I_n = (0, \frac{1}{n})$. However, $\cap_{i=1}^{\infty} I_i = \emptyset$ because there is no number x such that $0 < x < \frac{1}{n}$, $\forall n$.