Guangdong Technion
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Algebra A

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Tutorials and Workshops

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Week 1

Tutorials

1) Let $\mathbb{F}$ be a field. Prove the following properties:

(a) $a.0 = 0$ for every $a \in \mathbb{F}$.

(b) If $a.b = 0$ for $a, b \in \mathbb{F}$, then $a = 0$ or $b = 0$.

(c) $-(a+b) = -a + (-b)$ for every $a, b \in \mathbb{F}$.

(d) $(a.b)^{-1} = (b^{-1}).(a^{-1})$ for every $a, b \in \mathbb{F}$.

2) For an arbitrary set $\mathbb{F}$ of 4 elements find (if possible) addition and multiplication tables such that $\mathbb{F}$ is a field. Are there any other possible tables?

3) Consider the field $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

(a) Find $(1-\sqrt{2})^{-1}$.

(b) Calculate $\left(\frac{3}{2} + \sqrt{2}\right)^2 (1-\sqrt{2})^{-1} + \left(\frac{1}{2} + \frac{5}{4}\sqrt{2}\right)$.

4) Consider the fields $\mathbb{Z}_5$ and $\mathbb{Z}_{13}$.

(a) Find $4.(3^{-1}) + 2.(-6) + 3$ in $\mathbb{Z}_5$.

(b) Find $4.(3^{-1}) + 2.(-6) + 3$ in $\mathbb{Z}_{13}$.

5) Roots of unity.

(a) Find the cube roots of unity, i.e. all complex numbers z such that $z^3 = 1$.

(b) If ω is a cube root of unity, find the value of ω^{67}.

(c) Find the sixth roots of unity, i.e. all complex numbers z such that $z^6 = 1$.

6) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & 6 \end{pmatrix} \in \mathrm{M}_{2\times 3}(\mathbb{Z}_7)$, $B = \begin{pmatrix} 1 & 5 \\ 2 & 0 \\ 4 & 1 \end{pmatrix} \in \mathrm{M}_{3\times 2}(\mathbb{Z}_7)$ and $C = \begin{pmatrix} 23 & 65 \\ 14 & 6 \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{Z}_7)$.
Calculate $A.B + C$.

7) Show that $A = \begin{pmatrix} 4 & 5 \\ 1 & 6 \end{pmatrix}$ has an inverse in $\mathrm{M}_{2\times 2}(\mathbb{R})$ and write down the inverse explicitly.

8) Find the values of $a, b \in \mathbb{R}$ such that the remainder of the polynomial division of $p(x) = 3x^3 + 4x^2 - 2ax + b$ by $q(x) = x^2 + x + 1$ is $r(x) = -4x + 2$.

9) Let $p(x) = x^5 + x^4 + 2x^2 + 1$ and $q(x) = 2x^3 + x^2 + 1$. Find the remainder of the polynomial division of $p(x)$ by $q(x)$ in $\mathbb{Z}_3[x]$.

10) Show that $p(x) = x^3 + x^2 + x + 1$ is divisible by $q(x) = x + 3$ in $\mathbb{Z}_5[x]$ but not in $\mathbb{Z}[x]$.

Workshop

1) Calculate $(1+i)^{-5}$.

2) Let $A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$. Find a 2×2 matrix B such that $AB = 0$.

3) Find the inverse of the matrix $A = \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{Z}_{11})$.

4) Find all the roots of $p(x) = 5x^4 - 15x^2 - 20$ and their multiplicities.

WEEK 2

TUTORIALS

1) Check whether the following sets V with the defined operations are vector spaces over $\mathbb{F}$ or not. If not, what property fails?

(a) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \colon x \geq 0, y \geq 0 \right\}$ and $\mathbb{F} = \mathbb{R}$ with $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} x+w \\ y+z \end{pmatrix}$ and $c.\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$.

(b) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \colon xy \geq 0 \right\}$ and $\mathbb{F} = \mathbb{R}$ with $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} x+w \\ y+z \end{pmatrix}$ and $c.\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}$.

(c) Let $V = \mathrm{M}_n(\mathbb{R})$ and $\mathbb{F} = \mathbb{R}$ with $A + B = \frac{1}{2}(AB + BA)$ and $\alpha.A = 0$.

(d) Let X be an arbitrary set and $\mathcal{P}(A) = \{A \colon A \subseteq X\}$ the power set of X. Consider $V = \mathcal{P}(A)$ and $\mathbb{F} = \mathbb{Z}_2$ with $A + B = (A \setminus B) \cup (B \setminus A)$ and $0.A = \emptyset$, $1.A = A$.

(e) Let $V = \emptyset$ and any field $\mathbb{F}$ with any operations.

2) In the definition of a vector space V, show that the additive inverse condition can be replaced with the condition that $0.v = 0$ for all $v \in V$.

3) Consider the real vector space $\mathbb{R}^3$ with the usual operations.

(a) Is it possible to write the vector $0 = (0,0,0) \in \mathbb{R}^3$ in the form

$$0 = a.(2,2,2) + b.(0,0,3) + c.(0,1,1),$$

with $a, b, c \in \mathbb{R}$ not all zero?

(b) Is it possible to write any vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ uniquely in the form

$$v = a.(2,2,2) + b.(0,0,3) + c.(0,1,1),$$

with $a, b, c \in \mathbb{R}$?

(c) Is it possible to write the vector $u = (-1, 2, -1) \in \mathbb{R}^3$ in the form

$$u = a.(1,2,3) + b.(1,0,2),$$

with $a, b \in \mathbb{R}$?

(d) Is it possible to write any vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ uniquely in the form

$$v = a.(1,2,3) + b.(1,0,2) + c.(-1,2,-1),$$

with $a, b, c \in \mathbb{R}$?

4) Consider the real vector space given by $V = \{p \in \mathbb{R}[x] \colon \deg(p) \leq 2\}$ with the usual operations. We take $p_1(x) = 2 - x + 3x^2$, $p_2(x) = 1 + 4x + 2x^2$ and $p_3(x) = 8 - 10x + x^2$ polynomials in V. Is it possible to write the polynomial $0 \in V$ in the form

$$0 = a.p_1(x) + b.p_2(x) + c.p_3(x),$$

with $a, b, c \in \mathbb{R}$ not all zero?

5) Consider the vectors $u = (1, 2, 3)$ and $v = (2, 3, 1)$ in the real vector space $\mathbb{R}^3$.

(a) Find $k \in \mathbb{R}$ so that $w = (1, k, 4)$ is a linear combination of u and v.

(b) Find conditions on a, b, c so that $w = (a, b, c)$ is a linear combination of u and v.

6) Consider the real vector space $V = \{f \colon \mathbb{R} \to \mathbb{R} \colon f \text{ is a function}\}$ and the functions $cos, sin, id_{\mathbb{R}} \in V$. Show that if we have a linear combination of these elements in V equal to the zero function $0 \in V$ as in:

$$0 = a.cos + b.sin + c.id_{\mathbb{R}}$$

then $a = b = c = 0$ necessarily.

7) Consider the complex vector space $\mathbb{C}^3$ and $v_1 = (1, 1+i, i)$, $v_2 = (i, -i, 1-i)$ and $v_3 = (0, 1-2i, 2-i)$ vectors in $\mathbb{C}^3$.

(a) Is it possible to write one of these three vectors as a linear combination with coefficients in $\mathbb{R}$ of the other two? In other words, is it possible to write

$$0 = a.v_1 + b.v_2 + c.v_3,$$

with $a, b, c \in \mathbb{R}$ not all zero?

(b) Is it possible to write one of these three vectors as a linear combination with coefficients in $\mathbb{C}$ of the other two? In other words, is it possible to write

$$0 = a.v_1 + b.v_2 + c.v_3,$$

with $a, b, c \in \mathbb{C}$ not all zero?

8) Show that the following vector spaces share the same structure (i.e. their operations behave "the same"):

(a) $\mathrm{M}_2(\mathbb{R})$ and $\mathbb{R}^4$ as real vector spaces with the natural addition and scalar multiplication operations.

(b) $\mathbb{R}^4$ and $P_3 = \{p \in \mathbb{R}[x] \colon \deg(p) \leq 3\}$ as real vector spaces with the natural addition and scalar multiplication operations.

WORKSHOP

1) Let $V = \mathbb{R}^2$ and $\mathbb{F} = \mathbb{R}$. Consider the following operations on V:

$$(x, y) \oplus (w, z) = (x + w + 1, y + z + 1)$$
$$a \odot (x, y) = (ax, ay)$$

Which of the axioms for a vector space are satisfied by $(V, \oplus, \odot)$?

2) Let X be a set and $\mathbb{F}$ a field. Show that the set $V = \{f : X \to \mathbb{F} : f \text{ is a function}\}$ is a vector space over $\mathbb{F}$ with the addition $(f + g)(x) = f(x) + g(x)$ and the scalar multiplication $(af)(x) = af(x)$.

3) Consider V a vector space over the field $\mathbb{F}$. Prove the following statements:

(a) $-(-v) = v$ for every $v \in V$.

(b) $(-c).v = c.(-v) = -(c.v)$ for every $v \in V$ and $c \in \mathbb{F}$.

(c) If $v + w = u + w$ for some $v, w, u \in V$, then $v = u$.

4) Consider the real vector space $\mathrm{M}_2(\mathbb{R})$ with the usual operations. Is $A = \begin{pmatrix} 34 & 8 \\ 6 & -19 \end{pmatrix}$ a linear combination of $B = \begin{pmatrix} 5 & 1 \\ 0 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 1 \\ 2 & -3 \end{pmatrix}$?