Guangdong Technion
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Algebra A

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Tutorials and Workshops

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WEEK 1

TUTORIALS

1) Let $\mathbb{F}$ be a field. Prove the following properties:

(a) $a.0 = 0$ for every $a \in \mathbb{F}$.

(b) If $a.b = 0$ for $a, b \in \mathbb{F}$, then $a = 0$ or $b = 0$.

(c) $-(a + b) = -a + (-b)$ for every $a, b \in \mathbb{F}$.

(d) $(a.b)^{-1} = (b^{-1}).(a^{-1})$ for every $a, b \in \mathbb{F}$.

2) For an arbitrary set $\mathbb{F}$ of 4 elements find (if possible) addition and multiplication tables such that $\mathbb{F}$ is a field. Are there any other possible tables?

3) Consider the field $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

(a) Find $(1 - \sqrt{2})^{-1}$.

(b) Calculate $\left(\frac{3}{2} + \sqrt{2}\right)^2 (1 - \sqrt{2})^{-1} + \left(\frac{1}{2} + \frac{5}{4}\sqrt{2}\right)$.

4) Consider the fields $\mathbb{Z}_5$ and $\mathbb{Z}_{13}$.

(a) Find $4.(3^{-1}) + 2.(-6) + 3$ in $\mathbb{Z}_5$.

(b) Find $4.(3^{-1}) + 2.(-6) + 3$ in $\mathbb{Z}_{13}$.

5) Roots of unity.

(a) Find the cube roots of unity, i.e. all complex numbers z such that $z^3 = 1$.

(b) If ω is a cube root of unity, find the value of ω^{67}.

(c) Find the sixth roots of unity, i.e. all complex numbers z such that $z^6 = 1$.

6) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 0 & 6 \end{pmatrix} \in \mathrm{M}_{2\times 3}(\mathbb{Z}_7)$, $B = \begin{pmatrix} 1 & 5 \\ 2 & 0 \\ 4 & 1 \end{pmatrix} \in \mathrm{M}_{3\times 2}(\mathbb{Z}_7)$ and $C = \begin{pmatrix} 23 & 65 \\ 14 & 6 \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{Z}_7)$. Calculate $A.B + C$.

7) Show that $A = \begin{pmatrix} 4 & 5 \\ 1 & 6 \end{pmatrix}$ has an inverse in $\mathrm{M}_{2\times 2}(\mathbb{R})$ and write down the inverse explicitly.

8) Find the values of $a, b \in \mathbb{R}$ such that the remainder of the polynomial division of $p(x) = 3x^3 + 4x^2 - 2ax + b$ by $q(x) = x^2 + x + 1$ is $r(x) = -4x + 2$.

9) Let $p(x) = x^5 + x^4 + 2x^2 + 1$ and $q(x) = 2x^3 + x^2 + 1$. Find the remainder of the polynomial division of $p(x)$ by $q(x)$ in $\mathbb{Z}_3[x]$.

WORKSHOP

1) Calculate $(1+i)^{-5}$.

2) Let $A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$. Find a 2×2 matrix B such that $AB = 0$.

3) Find the inverse of the matrix $A = \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \in \mathrm{M}_{2\times 2}(\mathbb{Z}_{11})$.

3) Find all the roots of $p(x) = 5x^4 - 15x^2 - 20$ and their multiplicities.