HOMEWORK 9

1) (30pts) Suppose V is finite-dimensional and Γ is a subspace of V^*. Show that

$$\Gamma = \{v \in V \colon \ \varphi(v) = 0 \text{ for every } \varphi \in \Gamma\}^0.$$

2) (20pts) Suppose V and W are finite-dimensional. Let $T \in \mathcal{L}(V, W)$ and suppose there exists $\varphi \in V^*$ such that $\text{Null}(T^*) = \langle \varphi \rangle$. Prove that $\text{Range}(T) = \text{Null}(\varphi)$.

3) (30pts) Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by $T(x, y) = (3x + 4y - z, x + y + z, -3x + 6y)$ and consider the bases of $\mathbb{R}^3$ given by $\mathcal{B} = \{(1, 1, 0), (0, 1, 1), (1, 0, 0)\}$ and $\mathcal{B}' = \{(1, -2, 1), (2, -3, 3), (-2, 2, -3)\}$

(a) Find the matrix $[T]_{\mathcal{B}'}$ of T in the basis $\mathcal{B}'$.

(b) Write $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P$ where P is an invertible matrix.

4) (20pts) Let V be finite dimensional, $\mathcal{B}$ a basis of V and $A \in \text{M}_n(\mathbb{F})$ an invertible matrix. Prove that there exist bases $\mathcal{B}_1$ and $\mathcal{B}_2$ of V such that:

(a) A is the change of basis matrix from $\mathcal{B}_1$ to $\mathcal{B}$.

(b) A is the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}_2$.