Guangdong Technion
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ALGEBRA A

WINTER 2024

HOMEWORK 8

1) (20 pts) Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis for $\mathbb{C}^3$ defined by

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (2, 2, 0).$$

Find the dual basis of $\mathcal{B}$.

2) (20pts) Suppose $V_1, \ldots, V_n$ are vector spaces. Prove that $(V_1 \times \cdots \times V_n)^*$ and $V_1^* \times \cdots \times V_n^*$ are isomorphic vector spaces.

3) (20pts) Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that there exists a subspace W of V such that $\dim(W) = \dim(V/U)$ and $V = U \oplus W$.

4) (10pts) Let $\mathbb{F}$ be a field and let $f\colon \mathbb{F}^2 \to \mathbb{F}$ be the linear functional defined by $f(x_1, x_2) = ax_1 + bx_2$. For each of the following linear operators T, find $g(x_1, x_2)$ where $g = T^* f$.

(a) $T(x_1 x_2) = (x_1, 0)$.

(b) $T(x_1, x_2) = (-x_2, x_1)$.

(c) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2)$.

5) (30pts) Consider the real vector space of $n \times n$ matrices $V = \mathrm{M}_n(\mathbb{F})$.

(a) If $B \in V$ is a fixed matrix, define a function f_B on V by $f_B(A) = \mathrm{trace}(B^T A)$. Show that f_B is a linear functional on V.

(b) Show that every linear functional on V is of the above form, i.e., is f_B for some B.

(c) Show that $\Phi\colon V \to V^*$, $\Phi(B) = f_B$ is an isomorphism.