HOMEWORK 7

1) (30 pts) Consider the linear map given by $T\colon \mathrm{P}_2(\mathbb{R}) \to \mathrm{P}_1(\mathbb{R})$, $T(ax^2+bx+c) = (a+b)x + 2c - a$.

(a) Let $\mathcal{C} = \{x, x^2, 1\}$ and $\mathcal{C}' = \{1, x\}$. Find the matrix of T with respect to $\mathcal{C}$ and $\mathcal{C}'$.

(b) Let $\mathcal{B} = \{x^2+1, x^2+x+1, x^2\}$ and $\mathcal{B}' = \{1, x-2\}$. Find the matrix of T with respect to $\mathcal{B}$ and $\mathcal{B}'$.

(c) Give a basis for $\mathrm{Null}(T)$ and $\mathrm{Range}(T)$.

2) (30pts) Let $T\colon V \to W$ be a linear map. Prove the following:

(a) If $T(v) = 0$ for all $v \in V$, then for any bases $\mathcal{B}_V$ and $\mathcal{B}_W$ of V and W respectively, the matrix of T with respect to $\mathcal{B}_V$ and $\mathcal{B}_W$ is the zero matrix.

(b) If $\mathrm{Null}(T)$ is non-trivial then there exists a basis $\mathcal{B}_V$ of V such that for any basis $\mathcal{B}_W$ of W the matrix of T with respect to $\mathcal{B}_V$ and $\mathcal{B}_W$ has at least one zero column. Moreover, one can take $\mathcal{B}_V$ in such a way to have $\dim(\mathrm{Null}(T))$ zero columns.

(c) There exist bases $\mathcal{B}_V$ and $\mathcal{B}_W$ of V and W respectively such that the matrix of T with respect to $\mathcal{B}_V$ and $\mathcal{B}_W$ is $\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$ where I_m is the $m \times m$ identity matrix and $m = \dim(\mathrm{Range}(T))$.

3) (20pts) Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V, V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

4) (20pts) Prove that every linear map from $\mathrm{M}_{n\times 1}(\mathbb{F})$ to $\mathrm{M}_{m\times 1}(\mathbb{F})$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathrm{M}_{n\times 1}(\mathbb{F}), \mathrm{M}_{m\times 1}(\mathbb{F}))$, then there exists an $m \times n$ matrix A such that $T(x) = Ax$ for every $x \in \mathrm{M}_{n\times 1}(\mathbb{F})$.