Homework 2

1) (40 pts) Check whether the following sets V with the defined operations are vector spaces over $\mathbb{F}$ or not. If they are vector spaces, prove the properties. If not, what property fails?

(a) The set $V = \{f\colon \mathbb{R} \to \mathbb{R}\colon f \text{ is an even function}\}$ and $\mathbb{F} = \mathbb{R}$ with the addition $(f+g)(x) = f(x) + g(x)$ and the scalar multiplication $(af)(x) = af(x)$.

(Remember that an even function is a function $f\colon \mathbb{R} \to \mathbb{R}$ which satisfies $f(t) = f(-t)$ for all $t \in \mathbb{R}$).

(b) The set $V = \{A \in \mathrm{M}_n(\mathbb{R})\colon A \text{ is invertible}\}$ and $\mathbb{F} = \mathbb{R}$ with the natural addition $A + B$ of matrices and the natural scalar multiplication $c.A$ (i.e. coordinate by coordinate).

(c) Let $V = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ and $\mathbb{F} = \mathbb{R}$. We consider the addition and multiplication of two real numbers as usual and for $x \in \mathbb{R}$ we define:

$$x + \infty = \infty + x = \infty, \qquad x + (-\infty) = (-\infty) + x = -\infty,$$
$$\infty + \infty = \infty, \qquad -\infty + (-\infty) = -\infty, \qquad \infty + (-\infty) = 0,$$

and

$$x.\infty = \begin{cases} -\infty & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ \infty & \text{if} \quad x > 0, \end{cases} \qquad x.(-\infty) = \begin{cases} \infty & \text{if} \quad x < 0, \\ 0 & \text{if} \quad x = 0, \\ -\infty & \text{if} \quad x > 0. \end{cases}$$

2) (10 pts) Is $\mathbb{R}$ a vector space over $\mathbb{C}$? Is $\mathbb{C}$ a vector space over $\mathbb{R}$? Explain.

3) (30 pts) Consider the real vector space $\mathrm{M}_2(\mathbb{R})$ with the usual operations.

(a) Is it possible to write the zero matrix $0 \in \mathrm{M}_2(\mathbb{R})$ in the form

$$0 = a.\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + b.\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + c.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with $a, b, c \in \mathbb{R}$ not all zero?

(b) Is it possible to write any matrix $A \in \mathrm{M}_2(\mathbb{R})$ uniquely in the form

$$A = a.\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} + b.\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} + c.\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with $a, b, c \in \mathbb{R}$?

4) (20 pts) Consider the real vector space $\mathbb{R}[x]$ with the usual operations. Is $q(x) = 3x^3 - 2x^2 + 7x - 8$ a linear combination of $r(x) = x^3 - 2x^2 - 5x - 3$ and $s(x) = 3x^3 - 5x^2 - 4x - 9$?