HOMEWORK 12

1) (30pts) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z, w) = (4x+7z+w, 7x+2y+11z+3w, x+2z+w, -4x-8z-w)$. Find a basis $\mathcal{B}$ of $\mathbb{R}^3$ such that $[T]_{\mathcal{B}}$ is upper triangular.

2) (30pts) Decide whether or not the following linear maps $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ are diagonalizable and if so, find a matrix P such that $P^{-1}[T]_{\mathcal{C}}P$ is a diagonal matrix.

(a) $T(x, y, z) = (-x - 2y + 2z, 4x + 3y - 4z, -2y + z)$.

(b) $T(x, y, z, w) = (2x - y, x + 4y, z + 3w, z - w)$.

(c) $T(x, y, z, w, v) = (3x + 2y + 4z, 2x + 2z, 4x + 2y + 3z, 3u + v, 2u + 2v)$.

3) (20pts) Let $A \in \mathrm{M}_n(\mathbb{F})$.

(a) Suppose A is *nilpotent*, that is, there exists some $m \in \mathbb{N}$ such that $A^m = 0$. Prove that 0 is the only eigenvalue of A.

(b) Suppose $A \neq 0$ is *idempotent*, that is, $A^2 = A$. Prove that 0 and 1 are the only eigenvalues of A.

4) (20pts) Suppose $T \in \mathcal{L}(\mathbb{C}^3, \mathbb{C}^3)$ is such that 6 and 7 are eigenvalues of T. Furthermore, suppose $[T]_{\mathcal{B}}$ is not a diagonal matrix with respect to any basis $\mathcal{B}$ of $\mathbb{C}^3$. Prove that there exists $(x, y, z) \in \mathbb{C}^3$ such that $T(x, y, z) = (17 + 8x, \sqrt{5} + 8y, 2\pi + 8z)$.